106. *Boolean Valued Analysis and Type I AW*-Algebras*

By Masanao OZAWA
Department of Information Sciences, Tokyo Institute of Technology

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1983)

1. Introduction. The structure theory of type I AW^*-algebras was instituted by Kaplansky [3] as a purely algebraic generalization of the theory of type I von Neumann algebras. However, his theory was not completed as he stated [3; p. 460], "One detail has resisted complete solution thus far: the uniqueness of the cardinal number attached to a homogeneous AW^*-algebra of type I." The above cardinal uniqueness problem has been open for 30 years (cf. [1; pp. 88, 111, 118 Exercise 10]) and Kaplansky [4; p. 843] conjectured that the answer is negative.

In this note, we shall outline a negative answer to this problem. Our method is due to Boolean valued analysis recently developed by Takeuti [7]–[9] and Ozawa [5], and the construction of the counterexample of the cardinal uniqueness problem will be reduced to P. J. Cohen's forcing argument (cf. [2], [10]) developed in the field of mathematical logic. Our argument also includes a complete classification of type I AW^*-algebras in terms of the cardinal numbers in Scott-Solovay's Boolean valued universe of set theory (cf. [10]). The proofs of the results in this note will be published in the forthcoming paper [6] with more detailed treatment. For the terminology and the basic theory of AW^*-algebras we shall refer to Berberian [1].

2. Boolean valued universe of sets. Let B be a complete Boolean algebra. *Scott-Solovay's Boolean valued universe* $V^{(B)}$ is defined by $V^{(B)}=\bigcup_{\alpha\in On} V_\alpha^{(B)}$, where $V_\alpha^{(B)}$ is defined by transfinite induction as follows: $V_0^{(B)}=\emptyset$ and

$$V_\alpha^{(B)}=\{u \mid u: \mathrm{dom}\,(u)\to B \quad \text{and} \quad \mathrm{dom}\,(u)\subseteq \textstyle\bigcup_{\beta<\alpha} V_\beta^{(B)}\}.$$

For any $u, v \in V^{(B)}$, the Boolean values $\|u\in v\|$ and $\|u=v\|$ are defined (cf. [10; § 13]), and then we define the Boolean value $\|\varphi(a_1, \cdots, a_n)\|$ for any formula φ of set theory with $a_1, \cdots, a_n \in V^{(B)}$ in the obvious way. There is a canonical embedding $u\to\check{u}$ of the universe V of sets into $V^{(B)}$ such that $\|\check{u}\in\check{v}\|$ ($\|\check{u}=\check{v}\|$) equals 1 if $u\in v$ ($u=v$) and equals 0 otherwise. The basic principles of Boolean valued analysis is the following transfer principle.

Theorem 1 (Scott-Solovay, cf. [10]). *If φ is a theorem of ZFC then $\|\varphi\|=1$ is also a theorem of ZFC.*

3. Hilbert spaces in $V^{(B)}$. We define real numbers as Dedekind cuts of rational numbers. Let

$$C^{(B)}=\{a\in V^{(B)}\mid \|a \text{ is a complex number}\|=1\},$$

and

$$C^{(B)}_{\infty}=\{a\in C^{(B)}\mid \exists M\in \boldsymbol{R},\ \||a|<\check{M}\|=1\}.$$

Let Ω be the Stone representation space of B. Let $B(\Omega)$ be the *-algebra of complex valued Borel functions on Ω, $N(\Omega)$ be the ideal of $B(\Omega)$ consisting of functions vanishing outside a meager set. Let $C(\Omega)$ be the algebra of all complex valued continuous functions on Ω. Then by the similar argument as in [7; Chapter 2, § 2] we have the following identifications which preserves algebraic and order structure in the obvious way.

Theorem 2. *$C^{(B)}\cong B(\Omega)/N(\Omega)$ and $C^{(B)}_{\infty}\cong C(\Omega)$.*

Let Z be a commutative AW^*-algebra. Then the set B of all projections in Z forms a complete Boolean algebra and we have $Z\cong C(\Omega)$ where Ω is the Stone representation space of B (cf. [1; § 7]). An *AW^*-module* X over Z is a Z-module with Z-valued inner product $\langle\cdot,\cdot\rangle$ which satisfies some additional axioms (cf. [4]). A *base* for an AW^*-module X is a family $\{e_i\}$ such that (i) $\langle e_i, e_i\rangle=1$ for any i, (ii) $\langle e_i, e_j\rangle=0$ if $i\neq j$, (iii) for any $x\in X$, if $\langle x, e_i\rangle=0$ for all i then $x=0$. For a cardinal number α, an AW^*-module X is called *α-homogeneous* if it has a base with cardinality α. Let $l^2(S)$ be the set of square summable functions on a set S, i.e.,

$$l^2(S)=\{\xi\mid \xi: S\to \boldsymbol{C} \text{ and } \textstyle\sum_{s\in S}|\xi(s)|^2<\infty\}.$$

Consider l^2-spaces in $V^{(B)}$. For any $S\in V^{(B)}$, let

$$l^2(S)^{(B)}=\{\xi\in V^{(B)}\mid \|\xi\in l^2(S)\|=1\},$$

and let

$$l^2(S)^{(B)}_{\infty}=\{\xi\in l^2(S)^{(B)}\mid \exists M\in \boldsymbol{R},\ \|\textstyle\sum_{s\in S}|\xi(s)|^2<\check{M}\|=1\}.$$

Then obviously $l^2(S)^{(B)}$ is a Hilbert space in $V^{(B)}$, i.e., $\|l^2(S)$ is a Hilbert space$\|=1$. For any $S\in V^{(B)}$, denote by $\operatorname{card}(S)_B$ the cardinality of S in $V^{(B)}$. The following theorem states that the class of all cardinal numbers in $V^{(B)}$ is a complete system of invariants of AW^*-modules over Z.

Theorem 3. (1) *For any $S\in V^{(B)}$, $l^2(S)^{(B)}_{\infty}$ is an AW^*-module over Z.* (2) *For any $S, S'\in V^{(B)}$, $l^2(S)^{(B)}_{\infty}\cong l^2(S')^{(B)}_{\infty}$ if and only if* $\operatorname{card}(S)_B=\operatorname{card}(S')_B$. (3) *For any AW^*-module X over Z, there is a unique cardinal number α in $V^{(B)}$* (i.e., $\|\alpha$ is a cardinal number$\|=1$) *such that $X\cong l^2(\alpha)^{(B)}_{\infty}$.*

4. A classification of type I AW^*-algebras. Denote by $\mathcal{L}(H)$ the algebra of all bounded operators on a Hilbert space H. Let H be a Hilbert space in $V^{(B)}$, i.e., $\|H$ is a Hilbert space$\|=1$. Let $\mathcal{L}(H)^{(B)}=\{x\in V^{(B)}\mid \|x\in \mathcal{L}(H)\|=1\}$ and let $\mathcal{L}(H)^{(B)}_{\infty}=\{x\in \mathcal{L}(H)^{(B)}\mid \exists M\in \boldsymbol{R},\ \|\|x\|<\check{M}\|=1\}$, where $\|x\|$ is the bound of an operator x. Let π be an automorphism of B. Then π can be extended to $\pi: V^{(B)} \to V^{(B)}$ such that for any formula $\varphi(a_1, \cdots, a_n)$ with $a_1, \cdots, a_n \in V^{(B)}$,

$$\|\varphi(\pi(a_1), \cdots, \pi(a_n))\| = \pi(\|\varphi(a_1, \cdots, a_n)\|)$$

(cf. [10; Theorem 19.3]). Two cardinal numbers α and β in $V^{(B)}$ are called *congruent* if there is an automorphism π of B such that $\|\alpha = \pi(\beta)\| = 1$. The following theorem gives a complete solution of the classification of type I AW^*-algebras.

Theorem 4. *Let Z be a commutative AW^*-algebra and B the complete Boolean algebra of projections in Z. Then we have the following:* (1) *For any non-zero Hilbert space H in $V^{(B)}$, $\mathcal{L}(H)_\infty^{(B)}$ is a type* I *AW^*-algebra with center isomorphic to Z.* (2) *For any type* I *AW^*-algebra A with center Z, there is a cardinal number α in $V^{(B)}$ such that $A \cong \mathcal{L}(l^2(\alpha))_\infty^{(B)}$.* (3) *For any $S, S' \in V^{(B)}$, $\mathcal{L}(l^2(S))_\infty^{(B)} \cong \mathcal{L}(l^2(S'))_\infty^{(B)}$ if and only if* card $(S)_B$ *and* card $(S')_B$ *are congruent.* (4) *An AW^*-algebra A is α-homogeneous if and only if $A \cong \mathcal{L}(l^2(\check{\alpha}))_\infty^{(B)}$.*

In the Boolean valued model theory, we say that cardinal numbers are *absolute* in $V^{(B)}$ whenever, $\|\check{\alpha}$ is a cardinal number$\| = 1$ if and only if α is a cardinal number. The following theorem is an immediate consequence of Theorem 4.

Theorem 5. *Let Z be a commutative AW^*-algebra and let B be the complete Boolean algebra of projections of Z. Then the cardinal numbers in $V^{(B)}$ are absolute if and only if for any homogeneous AW^*-algebra A with center isomorphic to Z there is a unique cardinal number α such that A is α-homogeneous.*

By the above theorem any complete Boolean algebra B for which the cardinal numbers are not absolute in $V^{(B)}$ yields a counterexample of the uniqueness of the cardinality attached to a homogeneous AW^*-algebra. By the method of forcing we have the following.

Theorem 6. *For any pair of infinite cardinal numbers α and β, there is an AW^*-algebra which is α-homogeneous and simultaneously β-homogeneous.*

Sketch of Proof. It is known [2; Lemma 19.9] that there is a notion of forcing $\langle P, \leq \rangle$ such that $M[G] \models \text{card}\,(\alpha^M) = \text{card}\,(\beta^M)$ for any standard transitive model M of ZFC and any generic filter G of P over M. Let B be the Boolean algebra of all regular open subsets of P. Then $\|\text{card}\,(\check{\alpha}) = \text{card}\,(\check{\beta})\| = 1$ in $V^{(B)}$. Thus by Theorem 4.(3), $\mathcal{L}(l^2(\check{\alpha}))_\infty^{(B)} \cong \mathcal{L}(l^2(\check{\beta}))_\infty^{(B)}$. This shows that an AW^*-algebra $\mathcal{L}(l^2(\check{\alpha}))_\infty^{(B)}$ is α-homogeneous and simultaneously β-homogeneous. Q.E.D.

References

[1] Berberian, S. K.: Baer *-Rings. Springer, Berlin (1972).
[2] Jech, T.: Set Theory. Academic Press, New York (1978).

[3] Kaplansky, I.: Algebras of type I. Ann. of Math., **56**, 460–472 (1952).
[4] ——: Modules over operator algebras. Amer. J. Math., **75**, 839–858 (1953).
[5] Ozawa, M.: Boolean valued interpretation of Hilbert space theory. J. Math. Soc. Japan, **35**, 609–627 (1983).
[6] ——: A classification of type I AW^*-algebras and Boolean valued analysis (preprint).
[7] Takeuti, G.: Two Applications of Logic to Mathematics. Iwanami and Princeton University Press, Tokyo and Princeton (1978).
[8] ——: A transfer principle in harmonic analysis. J. Symbolic Logic, **44**, 417–440 (1979).
[9] ——: Von Neumann algebras and Boolean valued analysis. J. Math. Soc. Japan, **35**, 1–21 (1983).
[10] Takeuti, G. and Zaring, W. M.: Axiomatic Set Theory. Springer, Heidelberg (1973).