100. Microlocal Study of Sheaves. I
Contact Transformations

By Masaki KASHIWARA*) and Pierre SCHAPIRA**)

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0. Introduction. In [2] we defined the micro-support of a complex of sheaves F on a real manifold X and studied its functorial properties. With this tool, we are now able to quantize contact transformations for any sheaves. We prove that such q.c.t. commute with the Sato microlocalization, and when the manifolds are complex analytic we prove that the structure sheaf $\mathcal{O}_X$ is invariant by q.c.t.

1. Let X be a real manifold of class C^α ($2\leq\alpha\leq\infty$, or $\alpha=\omega$), T^*X its cotangent bundle, and π the projection from T^*X to X. Let A be a commutative ring. We denote by $D^+(X)$ (resp. $D^b(X)$) the full subcategory of the derived category of complexes of sheaves of A-modules on X whose cohomology is bounded from below (resp. bounded).

Let $F\in Ob(D^+(X))$. The micro-support of F, $SS(F)$, is a closed conic subset of T^*X defined in [2]. Let Ω be a subset of T^*X. We set:

$$\mathcal{E}(\Omega)=\{F\in Ob(D^+(X))\,;\,SS(F)\cap\Omega=\emptyset\}.$$

Let $S(\Omega)$ be the set of morphisms in $D^+(X)$, $u\colon F\to G$, such that the mapping cone of u belongs to $\mathcal{E}(\Omega)$. Then $S(\Omega)$ satisfies the axioms of [1], which enable us to localize $D^+(X)$ by $S(\Omega)$. We denote by $D^+(X,\Omega)$ the triangulated category so constructed (for $p\in T^*X$ we write $D^+(X,p)$ instead of $D^+(X,\{p\})$).

Let q_j be the j-th projection from $X\times X$, ($j=1,2$) and let Δ be the diagonal of $X\times X$. For F and $G\in Ob(D^+(X))$, we define:

$$\mu\hom(F,G)=\mu_\Delta(\boldsymbol{R}\,\mathcal{H}om\,(q_2^{-1}F,\,q_1^!F)).$$

Recall that for a submanifold $Y\subset X$, $\mu_Y(*)$ is the functor of the Sato microlocalization along Y([4]). Thus $\mu\hom(F,G)$ is a complex of sheaves on $T^*X\simeq T^*_\Delta(X\times X)$,

Proposition 1. *Let $p\in T^*X$. Then there exists a natural isomorphism*:

$$\mathrm{Hom}_{D^+(X,p)}(F,G)\simeq\mathcal{H}^0(\mu\hom(F,G))_p.$$

2. Let (E,σ) be a real symplectic vector space, and λ_1, λ_2, λ_3 three Lagrangian planes in E. Let q be the quadratic form on $\lambda_1\oplus\lambda_2\oplus\lambda_3$ given by:

*) Research Institute for Mathematical Sciences, Kyoto University.
**) Université de Paris-Nord.

$$q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1).$$

By definition (cf. [3]) the index $\tau(\lambda_1, \lambda_2, \lambda_3)$ is the signature of q.

Now to a real function φ on X we associate the (non homogeneous) Lagrangian manifold $\Lambda_\varphi = \{(x, d\varphi(x))\}$. We say that φ is transversal to a Lagrangian manifold Λ at $p \in \Lambda$ if $\varphi(\pi(p)) = 0$ and if Λ and Λ_φ intersect transversally at p.

Set $\lambda_\Lambda(p) = T_p(\Lambda)$, $\lambda_\varphi(p) = T_p\Lambda_\varphi$, $\lambda_0(p) = T_p(\pi^{-1}\pi(p))$.

Lemma 2.1. *Let Λ be a* (conic) *Lagrangian manifold, $p \in \Lambda$, $F \in Ob(D^+(X))$ such that $SS(F) \subset \Lambda$ in a neighborhood of p. Let φ be a function transversal to Λ at p and j a number with $j \equiv (1/2)(\dim X + \dim(\lambda_0(p) \cap \lambda_\Lambda(p))) \bmod \boldsymbol{Z}$. Set $\tau_\varphi(p) = \tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_\varphi(p))$. Then $(\mathcal{H}^{\tau_\varphi(p)/2+j}_{\{\varphi \geq 0\}}(F))_{\pi(p)}$ does not depend on φ.*

Definition 2.2. Let M be an A-module. In the preceding situation we say that F is pure with shift d of type M at p (along Λ) if

$$(\mathcal{H}^j_{\{\varphi\geq 0\}}(F))_{\pi(p)} = M \qquad \text{for } j = -d + \frac{1}{2}(\dim X + \tau_\varphi(p))$$

$$= 0 \qquad \text{for } j \neq -d + \frac{1}{2}(\dim X + \tau_\varphi(p)).$$

If M is a free A-module of rank one, we say that F is simple.

Example. Let Y be a submanifold of X. Then the constant sheaf $\underline{A}_Y$ is simple with shift codim $Y/2$ along $T^*_Y X$. Moreover such a property characterizes $\underline{A}_Y$ in $D^+(X, p)$ for any $p \in T^*_Y X$.

Let Y and X be two C^2-manifolds, f a map from Y to X, and let ρ and $\tilde{\omega}$ be the natural associated maps from $Y \times_X T^*X$ to T^*Y and T^*X, respectively.

Proposition 2.3. *Let Λ be a Lagrangian manifold in T^*Y, $p \in Y \times_X T^*X$, and let $G \in Ob(D^+(Y))$. Assume:*

i) *ρ is transversal to Λ at p and $\tilde{\omega}$ is an immersion on $\rho^{-1}(\Lambda)$.*

ii) *$\rho^{-1}(SS(G)) \cap \tilde{\omega}^{-1}(\tilde{\omega}(p)) \subset \{p\}$.*

iii) *G is pure of type M along Λ at $\rho(p)$ with shift d.*

iv) *the map f is proper on* Supp(G).

Then $\boldsymbol{R}f_*(G)$ is pure of type M along $\Lambda_0 = \tilde{\omega}\rho^{-1}(\Lambda)$ at $\tilde{\omega}(p)$, with shift $d - (1/2)(\dim Y - \dim X) - (1/2)\tau(\rho\tilde{\omega}^{-1}(\lambda_0(\tilde{\omega}(p))), \lambda_0(\rho(p)), \lambda_\Lambda(\rho(p)))$.

Proposition 2.4. *Let Λ be a Lagrangian manifold in T^*X, $F \in Ob(D^+(X))$ and $p \in Y \times_X \Lambda$. Assume;*

i) *$\tilde{\omega}$ is transversal to Λ and ρ is an immersion on $\tilde{\omega}^{-1}(\Lambda)$.*

ii) *$\tilde{\omega}^{-1}(SS(F)) \cap \rho^{-1}\rho(p) \subset \{p\}$.*

iii) *F is pure of type M with shift d along Λ at p.*

iv) *f is non characteristic for F, that is $T^*_Y X \cap \rho^{-1}(SS(F)) \subset Y \times_X T^*_X X$.*

Then $f^{-1}(F)$ is pure of type M along $\rho(\tilde{\omega}^{-1}(\Lambda))$ at $\rho(p)$ with shift d.

3. Let q_1 and q_2 be the projections from $X\times Y$ to X and Y, and let p_1 and p_2 be the projections from $T^*(X\times Y)$ to T^*X and T^*Y, respectively. Set $p_j^a = p_j \circ a$, where a is the anti-podal map. Let Λ be a Lagrangian manifold in $T^*(X\times Y)$ and assume that p_1^a is an immersion at $\lambda\in\Lambda$ (thus p_2 is a submersion). Let K be a simple sheaf along Λ at λ. Then we can construct $K'\in Ob(D^b(X\times Y))$, with $K'\cong K$ in $D^+(X\times Y, \lambda)$ such that:

i) $SS(K')\cap(p_1^a)^{-1}p_1^a(\lambda)\subset\{\lambda\}$

ii) $SS(K')\cap(p_1^a)^{-1}(T_X^*X)\subset T_{X\times Y}^*(X\times Y)$

iii) q_1 is proper over supp (K').

Then we define the functor φ_K from $D^+(Y, p_2(\lambda))$ to $D^+(X, p_1^a(\lambda))$ by setting for $G\in Ob(D^+(Y))$:

$$\varphi_K(G) = \boldsymbol{R}q_{1*}\boldsymbol{R}\,\mathcal{H}om\,(K', q_2^!G).$$

Theorem 3.1. *Assume $p_1^a|_\Lambda$ and $p_2|_\Lambda$ are local isomorphisms at λ. Then the functor φ_K defines an equivalence of category between $D^+(Y, p_2(\lambda))$ and $D^+(X, p_1^a(\lambda))$. Moreover for F, $G\in D^+(Y)$ we have:*

$$\mu\,\mathrm{hom}\,(\varphi_K(F), \varphi_K(G))\cong\varphi_*\mu\,\mathrm{hom}\,(F, G)$$

in a neighborhood of λ. Here $\varphi=(p_1^a|_\Lambda)\circ(p_2|_\Lambda)^{-1}$.

Along any Lagrangian manifold there exist microlocally simple sheaves, and such sheaves are unique up to the shift. Thus any homogeneous symplectic transformation φ may be "quantized", that is extended to an isomorphism φ_K as in Theorem 3.1, and such a quantization is unique up to the shift. Of course $SS(\varphi_K(G))=\varphi(SS(G))$, in a neighborhood of $p_1^a(\lambda)$, and this is in fact the key of our proof of the microsupport of sheaves. Moreover these quantizations commute with microlocalization, since for a submanifold $M\subset X$, and $F\in Ob(D^+(X))$, we have:

$$\mu_M(F)=\mu\,\mathrm{hom}\,(\underline{A}_M, F)$$

where $\underline{A}_M$ is the constant sheaf on M.

4. We assume now X and Y complex analytic, and denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X ($\mathcal{O}_Y$ on Y).

Theorem 4.1. *In the situation of Theorem 3.1 with Λ a complex Lagrangian manifold, we have a natural isomorphism*

$$\varphi_K(\mathcal{O}_Y)\cong\mathcal{O}_X \qquad \text{in } D^+(X, p_1^a(\lambda)).$$

We may apply Theorem 4.1, Propositions 2.3 and 2.4 to calculate the cohomology groups $\mathcal{H}_M^j(\mathcal{O}_X)$ for real submanifolds M of X, since φ_K commutes with microlocalization.

References

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