97. On v-Ideals in a VHC Order*

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Throughout this note, Q will be a simple artinian ring and R will be an order in Q with 1. Let $\mathcal{C}$ ($\mathcal{C}'$) be a right (left) Gabriel topology on R cogenerated by the right (left) injective hull of $Q/R$. In [4], $R$ is called a VH ($v$-hereditary) order if for any $R$-ideal $A$ such that $\mathcal{v}A = A$ we have $\mathcal{v}(A(R:A)_v) = O_i(A)$ (resp. $(R:A)_v = O_i(A)$). We say that $R$ is a VHC order if it is a VH order satisfying the maximum condition on $\mathcal{C}$-closed right ideals and $\mathcal{C}'$-closed left ideals. The concept of VHC orders is a Krull type generalization of HNP (hereditary noetherian prime) rings. The aim of this note is to extend Robson's theorems and Fujita-Nishida's theorems in HNP rings to the case of VHC orders (cf. [1], [7] and [3]). Concerning our terminology and notations we refer to [4]. See [6] for many interesting examples of VHC orders.

Proposition 1. The following two conditions are equivalent:
(1) $\mathcal{v}(A(R:A)_v) = O_i(A)$ for any $R$-ideal $A$ such that $\mathcal{v}A = A$.
(2) $\mathcal{v}(A(R:A)_v) = (O_i(A))$ for any $R$-ideal $A$.

Proof. (2)$\Rightarrow$(1) is clear, because $(O_i(A)) = O_i(A)$ for any $R$-ideal $A$ with $\mathcal{v}A = A$. (1)$\Rightarrow$(2): Since $\mathcal{v}A \supset A$, we have $1 \in O_i(A) = \mathcal{v}(A(R:A)_v) \subset \mathcal{v}(A(R:A)_v) = (O_i(A))$ by Lemma 1.1 of [4]. It is clear that $A(R:A)_v \subset O_i(A)$ and so $\mathcal{v}(A(R:A)_v) \subset O_i(A))$. On the other hand, $A(R:A)_v$ is an $(O_i(A), O_i(A))$-bimodule and thus $\mathcal{v}(A(R:A)_v)$ is a right $O_i(A)$-module. Hence it follows that $O_i(A) \subset \mathcal{v}(A(R:A)_v)$ and that $\mathcal{v}(O_i(A)) \subset \mathcal{v}(A(R:A)_v)$.

From now on, $R$ will be a VHC order in a simple artinian ring Q.

Lemma 1. Let $A$ be any $R$-ideal. Then $\mathcal{v}A = A$.

Proof. This is proved as in Lemma 1.2 of [4] by using Proposition 1.

We consider the following sets of v-ideals of $R$: $V(R) = \{A: \text{ideal of } R \mid A: \text{v-ideal}\} \supseteq V_m(R) = \{A \in V(R) \mid A \subset P: \text{prime v-ideal} \Rightarrow P: \text{maximal v-ideal}\}$. If $R$ has enough v-invertible ideals, then $V(R) = V_m(R)$ by Lemma 1.2 of [5]. We do not have an example of VHC order in which $V(R) \supset V_m(R)$ up to now. We study the properties of ideals belonging to $V_m(R)$.

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Proposition 2. (1) If $A, B \in V_m(R)$, then $AB \in V_m(R)$.

(2) Let $A$ and $B$ be elements in $V(R)$ such that $A \subset B$. If $A \in V_m(R)$, then $B \in V_m(R)$.

(3) If $A \in V_m(R)$, then $Ass(R/A)$ consists of maximal $v$-ideals of $R$.

(4) Let $X$ be any $v$-invertible ideal of $R$. Then $X \in V_m(R)$.

(5) Let $A$ be any element in $V(R)$. Then $A \in V_m(R)$ if and only if there are maximal $v$-ideals $M_1, \ldots, M_n$ satisfying $M_1 \cdots M_n \subset A \subset M_i$ for any $i = 1, \ldots, n$.

Proof. (1), (2) and (3) are trivial. (4): As in Propositions 2.10 and 2.11 of [4], we have $R = \bigcap R_p \cap S(R)$, where $R_p$ is an HNP ring whose Jacobson radical $P' = PR_p = R_p P$ is a unique maximal invertible ideal of $R_p$ ($P$ ranges over all maximal $v$-invertible ideals of $R$), $S = S(R) = \bigcup Y^{-1}$ ($Y$ runs over all $v$-invertible ideals of $R$), and $(SX)_* = S = (SX)_*$. Now let $A$ be a prime $v$-ideal containing $X$. Then we have $A = \bigcap AR_p \cap (AS)_* = \bigcap AR_p \cap S$. There are only a finite number of maximal $v$-invertible ideals $P_1, \ldots, P_n$ of $R$ such that $R_{P_i} \supset A R_{P_i}$ ($1 \leq i \leq n$) and so $A = A_1 \cap \cdots \cap A_n (A_j = A R_{P_i} \cap R)$. Since $A$ is a prime ideal, we have $A = A_i$ for some $i$ and so $AR_{P_i}$ is also a prime ideal. Write $P_i = M_{i_1} \cap \cdots \cap M_{i_2}$, an intersection of a cycle, where $M_j$ are maximal $v$-ideals of $R$. Then $\{M_{i_j} R_{P_i} | 1 \leq j \leq k\}$ are only prime ideals of $R_{P_i}$ (see Proposition 2.7 of [4]). Thus $AR_{P_i} = M_{i_j} R_{P_i}$ for some $j$ and $A = AR_{P_i} \cap R = M_{i_j}$, a maximal $v$-ideal of $R$. Since $R$ satisfies a.c.c. on $v$-ideals of $R$, (5) easily follows (see the proof of Lemma 1.2 of [8]).

Proposition 3. (1) Let $A$ be any element in $V_m(R)$. Then $A = (XB)_*$ for some $v$-invertible ideal $X$ of $R$ and eventually $v$-idempotent ideal $B \in V_m(R)$.

(2) Let $C$ be an eventually $v$-idempotent ideal in $V_m(R)$ and let $M_1, \ldots, M_n$ be the full set of maximal $v$-ideals containing $C$. Then $(C^*_* = (M_1 \cap \cdots \cap M_n)^*_* = v$-idempotent.

Proof. (1) As in Theorem 4.2 of [1]. (2) follows from the proof of Proposition 1.4 of [6].

Lemma 2. Let $M_1$ and $M_2$ be any maximal $v$-ideals of $R$ such that $O_i(M_2) \neq O_i(M_2)$ for all $i$, $j$ ($1 \leq i, j \leq 2$) and let $A = M_1 \cap M_2$. Then $A = (M_1 \cap M_2)_* = (M_1 \cap M_2)$, and is $v$-idempotent.

Proof. First we note that $A \in V_m(R)$. Assume that $A$ is not $v$-idempotent. Then, by Lemma 1.3 of [6], we have $R \supseteq (A : R : A)_* \supseteq A$ and $R \supseteq ((R : A)_*)_* \supseteq A$, because $((R : A)_*)_*$ and $(A (R : A)_*)_*$ are both $v$-idempotent. So we may assume that $((R : A)_*)_* = M_1$ by Propositions 2 and 3, and then $A = (M_1 M_2)_*$ by Lemma 1.3 of [6]. Thus we have $O_i(A) = O_i(M_1)$. Assume that $M_1 = (A (R : A)_*)_*$. Then $O_i(M_2) \supseteq O_i(A) \supseteq O_i(M_1)$ and so $M_2 \subset M_1$. This is a contradiction. Hence $M_1 \supseteq M_2$.
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Now assume that $W = O_r(M_1) \cap O_r(M_2) \supset R$. Then $R \supseteq (R : W), \supseteq (R : O_r(M_1)) = M_1$, and so $(R : W)_r = M_1$. Similarly, we have $(R : W)_v = M_1$. Thus $O_r(M_1) = W = W = O_r(M_2)$ by Lemma 1. This is a contradiction. Hence $O_r(M_1) \cap O_r(M_2) = R$. On the other hand, since $(A^2)_r$ is $v$-idempotent by Lemma 1.3 of [6], we have $K = O_r((A^2)_r) \cap O_r((A^2)_v) \supseteq R$ by the same method as in Lemma 1.7 of [6]. The inclusions $(A^2)_v \subseteq (R : K) \subseteq R$ imply that $(R : K)_r$ is contained in a maximal $v$-ideal of $R$, say $M_1$. Then $K = K \supseteq O_r(M_1) \supseteq R$. This entails that $O_r(M_1)$ is a $v$-ideal. So it follows from Lemma 1.7 of [2] that there exists a $v$-idempotent ideal $N$ containing $(A^2)_r$ such that $O_r(M_1) = O_r(N)$. Since $O_r(M_1)$ is minimal in the set of all overrings of $R$ which are $v$-ideals, $N$ must be a maximal $v$-ideal of $R$ and thus $N = M_1$, which is a contradiction. Therefore $A$ must be $v$-idempotent.

Distinct $v$-idempotent, maximal $v$-ideals $M_1, \ldots, M_n$ are called an open cycle if $O_r(M_1) = O_r(M_2), \ldots, O_r(M_{n-1}) = O_r(M_n)$ but $O_r(M_n) \neq O_r(M_1)$. The following proposition is due to Fujita and Nishida if $R$ is an HNP ring which is obtained in a similar way to prove Theorem 1.3 of [3] by using Lemma 1.3 of [6], Propositions 2, 3 and Lemma 2.

Proposition 4. Let $M_1, \ldots, M_n$ be an open cycle and let $A = M_1 \cap \cdots \cap M_n$. Then

1. $(A : r) = M_i$ and $(R : A)_v = M_n$.
2. $A = (M_1 \cdots M_n)_r$.
3. $(AM_i)_v = (M_1 \cdots M_{i-1} A)_v$ for $i = 1, \ldots, n-1$.
4. $(A^\alpha (R : A)_r) = (M_1 \cdots M_n)_v$ and $((R : A)_r A^\alpha) = (M_1 \cdots M_{n-1})_v$.
   In particular, $(A^\alpha)_v = (A^\alpha (R : A)_r)_v = ((R : A)_r A^\alpha)_v = (M_1 \cdots M_n)_v$.
5. $A \supseteq (A^2)_r \supseteq \cdots \supseteq (A^n)_r = (A^{n+1})_v = \cdots$.

Let $M_1, \ldots, M_m$ and $N_1, \ldots, N_n$ be distinct $v$-idempotent, maximal $v$-ideals of $R$. Then, following [3], they are separated if $O_r(M_1) \neq O_r(N_j)$ and $O_r(N_j) \neq O_r(M_i)$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Proposition 3 allows us to study $v$-invertible ideals and eventually $v$-idempotent ideals separately. The structure of $v$-invertible ideals was completely determined in [4] (see Theorem 1.13 of [4]). To study eventually $v$-idempotent ideals of $R$, let $M_1, \ldots, M_n$ be a finite set of distinct $v$-idempotent, maximal $v$-ideals of $R$ such that $A = M_1 \cap \cdots \cap M_n$ is not contained in any $v$-invertible ideal of $R$ (see Proposition 3). Then we classify it as follows;

- (a) $\{M_i, \ldots, M_n\} = \bigcup_{i=1}^n \{M_{i1}, \ldots, M_{in} \}$, and each of $M_{i1}, \ldots, M_{in(t)}$ is an open cycle.
- (b) $M_{i1}, \ldots, M_{in(t)}$ and $M_{j1}, \ldots, M_{jn(t)}$ are separated for any $i, j$ ($i \neq j$). Put $A_i = M_{i1} \cap \cdots \cap M_{in(t)}$. Then we have

Proposition 5. With the above notations and assumption we
have \( A = (A_1 \cdots A_n) \) and \( (A_i A_j)_e = (A_j A_i)_e \) (cf. [3]).

Proof. By Proposition 4, \( A_i = (M_{1i} \cdots M_{ni})_e \) and so \( (A_i A_j)_e = (A_j A_i)_e \) by Lemma 2. We shall prove \( A = (A_1 \cdots A_n)_e \) by induction on \( k \). If \( k = 1 \), then there is nothing to prove. So we may assume that \( B = A_1 \cdots A_{k-1} = (A_1 \cdots A_{k-1})_e \). Then \( (BA_k)_e = (A_i B)_e \) by Lemma 2 and \( (B + A_k)_e = R \). Thus \( A = B \cap A_k = ((B \cap A_k)(B + A_k)_e)_e \subset (BA_k)_e \) and \( (A_i B)_e = (BA_i)_e = (A_1 \cdots A_i)_e \) and therefore \( A = (A_1 \cdots A_n)_e \).

The next proposition is due to Robson in case \( R \) is an HNP ring (see [7]) and the author obtained the proposition if \( R \) is a VHC order with enough \( v \)-invertible ideals (see [6]).

Proposition 6. Let \( M_1, \ldots, M_n \) be maximal \( v \)-ideals of \( R \) and let \( A = M_1 \cap \cdots \cap M_n \). Then \( A \) is \( v \)-idempotent if and only if \( O_i(M_i) \neq O_i(M_j) \) for any \( i, j \).

Proof. Assume that \( A \) is \( v \)-idempotent and that \( O_i(M_i) = O_i(M_j) \) for some \( i, j \). If \( i = j \), then \( M_i \) is \( v \)-invertible and so \( A \cap \cap_{k=1}^{n} (M_i)_e = O_i \), a contradiction. Hence \( i \neq j \). Let \( A = (A_1 \cdots A_n)_e \) be the decomposition of \( A \) as in Proposition 5. Then there exists \( A_i \), say \( A_1 \), such that \( A_1 = M_1 \cap \cdots \cap M_{1n(1)} \) with \( n(1) \geq 2 \). Then we have, by Proposition 4, \( s(M_{1n(1)}A_1 \cdots A_n) = s((R : A_1)A_1A_2 \cdots A_n) = (R : A_1)A_1^2A_2^2 \cdots A_n^2) = (M_{1n(1)}A_1^2 \cdots A_n^2) \subset M_{11} \), which is a contradiction. Hence \( O_i(M_i) \neq O_i(M_j) \) for all \( i, j \). We prove the sufficiency by induction on \( n \) (see Lemma 2 in case \( n = 2 \)). So we may assume that \( B = M_1 \cap \cdots \cap M_{n-1} = (M_1 \cdots M_{n-1})_e \) is \( v \)-idempotent and \( (B + M_n)_e = R \). Thus \( A = B \cap M_n = ((B \cap M_n)(B + M_n)_e)_e \subset ((BM_n)_e + (M_nB)_e)_e = (M_1 \cdots M_n)_e \) by Lemma 2. Hence \( A = (M_1 \cdots M_n)_e \) and is \( v \)-idempotent, because \( (M_i M_j)_e = (M_i M_j)_e \).

References