87. *Degeneration of Surfaces with Trivial Canonical Bundles*

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The purpose of this note is to study a degeneration of surfaces with trivial canonical bundles, especially one which contains a surface of class VII in its singular fiber. Details will be published elsewhere. I would like to thank Prof. K. Ueno for his invaluable suggestions and encouragement.

§1. Let $\pi: X\to\Delta$ be a proper surjective holomorphic map of a three dimensional complex manifold X to a disk $\Delta=\{t\in C\,|\,|t|<\varepsilon\}$ with connected fibers. Assume that π is smooth at each point of $\pi^{-1}(\Delta^*)$, $\Delta^*=\Delta-\{0\}$. We call such a holomorphic map $\pi: X\to\Delta$ a *degeneration of surfaces* (or briefly, a *degeneration*). By a singular fiber X_0, we mean a divisor on X defined by $\pi=0$. A smooth fiber $X_t=\pi^{-1}(t)$ $(t\neq 0)$ is called a general fiber.

A degeneration $\pi': X'\to\Delta$ is called a modification of a degeneration $\pi: X\to\Delta$, if there exists a bimeromorphic map $\Phi: X \dashrightarrow X'$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\Phi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & \Delta & \end{tikzcd}$$

is commutative and over Δ^*, res $\Phi: \pi^{-1}(\Delta^*)\to\pi'^{-1}(\Delta^*)$ is biholomorphic.

A degeneration $\pi: X\to\Delta$ is called semi-stable, if the singular fiber X_0 is a reduced divisor with simple normal crossings. Note that by Mumford's theorem every degeneration can be made semi-stable after a base change and a modification.

In this note, we shall study degenerations of surfaces up to modifications. We are mainly interested in a semi-stable degeneration of $K3$ surfaces which is not assumed to be projective nor Kähler.

§2. **Theorem.** *Let $\pi: X\to\Delta$ be a semi-stable degeneration of $K3$ surfaces. Then this satisfies one of the following conditions:*

(i) *there exists a modification $\pi': X'\to\Delta$ of $\pi: X\to\Delta$ such that π' is also semi-stable and the canonical bundle $K_{X'}$ on X' is trivial.*

(ii) *one of the components of the singular fiber X_0 is a Hopf surface of its blown-up surface.*

(iii) *one of the components of the singular fiber X_0 is a VII surface whose minimal model S has only finite number of curves which are non-singular rational and form just one cycle with some branches.*

To prove this, we use a result of Enoki [1].

Remark. (1) If every component of the singular fiber X_0 is algebraic (or Kähler), it satisfies (i). This is a result of Kulikov [6], [7] and Persson-Pinkham [9].

(2) There are examples which satisfy both (i) and (ii) (see Example 1 below). There are also examples which do not satisfy (i), but satisfy (ii). See § 3, Example 2 below.

(3) The cases (i) and (iii) are disjoint. There are examples of the case (iii). See § 4, Example 3 below.

(4) In the case (iii), a divisor of double curves on the VII_0 surface S is contractible to a normal Gorenstein singular point with geometric genus 2 (see § 5).

(5) The above theorem holds for a semi-stable degeneration of surfaces with trivial canonical bundles, that is, a general fiber may be a complex torus of dimension two or a Kodaira surface. But the author does not know any examples of the case (iii) with complex tori or Kodaira surfaces as general fibers.

Example 1. Ueno has constructed an example of a semi-stable degeneration of $K3$ surfaces whose singular fiber is $V_1 \cup_{E_1} V_2 \cup_{E_2} V_3$, where V_1 and V_3 are rational surfaces, V_2 is a Hopf surface, and E_1 and E_2 are elliptic curves. This example satisfies both (i) and (ii).

§ 3. Example 2. Let S be an elliptic $K3$ surface, and E a smooth fiber on S (an elliptic curve). Then we can construct a semi-stable degeneration $\pi: X \to \Delta$, whose general fiber $\pi^{-1}(t)$ $(t \neq 0)$ is an elliptic $K3$ surface (which is a logarithmic transform of S), and whose singular fiber $\pi^{-1}(0)$ is a union of the elliptic $K3$ surface S and an elliptic Hopf surface (which is a logarithmic transform of $E \times \mathrm{P}^1$). The tool of the construction is a "simultaneous logarithmic transformation".

§ 4. Example 3. Let $X_0 = V_0 + \cdots + V_6$ be a two dimensional compact analytic space with normal crossings whose configuration is as in Fig. 1. All components V_i's are smooth, and all double curves A_i's, B_i's, C_i's and D_i's are non-singular rational.

V_0 is a surface of class VII_0 with only three curves A_1, A_2 and A_3 whose self-intersection numbers on V_0 are -2, -2, and -3, respectively, and the canonical bundle K_{V_0} on V_0 is written as $K_{V_0} = -2A_1 - 4A_2 - 3A_3$. (See Kato [4] for the construction of such a surface V_0.) V_4 is a surface obtained by blowing up an elliptic $K3$ surface. The other components are all rational surfaces. We omit detailed description of them.

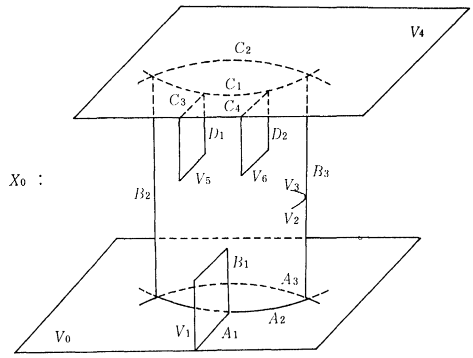

Fig. 1

Then, using the deformation theory due to Friedman [2], we can prove that the variety X_0 is smoothable to $K3$ surfaces, more precisely, there exists a semi-stable degeneration of $K3$ surfaces whose singular fiber is isomorphic to X_0.

§ 5. Let S be a VII_0 surface described in (iii) of Theorem. Then a divisor of all curves on S can be contracted to a point P on a normal surface S_0. We assume that $K_S=-D$, where D is an effective divisor. Then we have the following

Proposition. *Assume that the isolated singular point P is smoothable as a germ. Then the normal surface S_0 is also smoothable. Moreover, a general fiber of this smoothing of S_0 is a $K3$ surface.*

Example 4. Let S be a VII_0 surface as follows

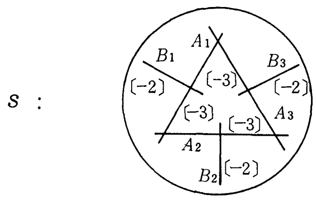

Fig. 2

where $[m]$ denotes the self-intersection number of each curve. K_S is written as $-D$, where $D=2A_1+2A_2+2A_3+B_1+B_2+B_3$. Then the above proposition is applicable to this surface S. (The smoothability of the singular point P is checked by using Yau's theorem in [10].) On the other hand, S can also be made a component of a singular fiber in a semi-stable degeneration of $K3$ surfaces, as in Example 3.

§ 6. **Example 5.** Using "logarithmic transformations" for a very special elliptic 3-fold, we can construct a degeneration $\pi: X \to \Delta$ of abelian surfaces whose singular fiber has a component of a surface with Kodaira dimension 1. More precisely, this degeneration is as follows. Over Δ^*, $\operatorname{res}\pi: \pi^{-1}(\Delta^*) \to \Delta^*$ is a trivial family of the product

$C\times E$ of two elliptic curves C, E. The singular fiber X_0 is a divisor with normal crossings and is written as

$$X_0=2S_0+\sum_{i=1}^{2r} 2S_i.$$

Here S_0 is an elliptic surface over C with $2r$ multiple fibers of type ${}_2I_0$, and is obtained from $C\times E$ by means of $2r$ logarithmic transformations. Clearly S_0 has Kodaira dimension 1. S_i $(i=1, \cdots, 2r)$ is an elliptic Hopf surface with a multiple fiber of type ${}_2I_0$, and is obtained from $P^1\times E$ by means of a logarithmic transformation. S_i and S_j $(i, j\geqq 1)$ are disjoint to each other. S_0 and S_i $(i\geqq 1)$ cross transversely along an elliptic curve which is a multiple fiber on the elliptic surfaces S_0, S_i.

Remark. There is also a semi-stable degeneration of $K3$ surfaces whose singular fiber has a component of a surface with Kodaira dimension 1 (see Nishiguchi [8]). The deformation theory due to Friedman [2] was used to construct it in [8], as in Example 3 of § 4.

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