83. *Variations of Pseudoconvex Domains*

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1. Let $\boldsymbol{R}^m$ be the real euclidean space of dimension m ($\geqq 2$) with norm $\|x\|^2=|x_1|^2+\cdots+|x_m|^2$, where $x=(x_1, \cdots, x_m)$ is the standard coordinate system. By an unramified covering domain over the space $\boldsymbol{R}^m$ or, more simply, a domain over $\boldsymbol{R}^m$, we mean a connected Hausdorff space E together with a locally homeomorphic map p of E to $\boldsymbol{R}^m$. If there is no ambiguity, we use the notation $x \in E$, which means precisely that x is a point of E such that $p(x)=x$ ($\in \boldsymbol{R}^m$). Now consider a domain D over $\boldsymbol{R}^m$ and fix a point x^o in D. We take a sequence of relatively compact subdomains D_p ($p=1, 2, \cdots$) of D such that $x^o \in D_1$, $D_p \subset D_{p+1}$, $\bigcup_{p=1}^{\infty} D_p=D$ and the boundaries ∂D_p of D_p in D are real analytic. According to the potential theory, every D_p carries the Green function $g_p(x)$ with pole x^o, which is uniquely determined by the following three conditions: $\Delta g_p=\partial^2 g_p/\partial x_1^2+\cdots+\partial^2 g_p/\partial x_m^2=0$ on $D_p-\{x^o\}$, $g_p(x)=0$ on ∂D_p, and on a neighborhood of x^o in D_p, $g_p(x)$ is expanded in the form

$$g_p(x)=\log \frac{1}{\|x-x^o\|}\left(\text{resp. } \frac{1}{\|x-x^o\|^{m-2}}\right)+\lambda_p+h_p(x)$$

for $m=2$ (resp. $m\geqq 3$), where λ_p is a constant, $h_p(x)$ is harmonic and $h_p(x^o)=0$. Since the functions $g_p(x)$ and the constants λ_p increase with p, the limits $g(x)=\lim_{p\to\infty} g_p(x)$ and $\lambda=\lim_{p\to\infty} \lambda_p$ exist. It is clear that $0<g(x)\leqq +\infty$ on D, $-\infty<\lambda\leqq +\infty$ (resp. $\leqq 0$) for $m=2$ (resp. $m\geqq 3$) and that $g(x)\equiv +\infty$ on D if and only if $\lambda=+\infty$ for $m=2$. This $g(x)$ is the Green function of D with pole x^o, and the constant term λ is called the *Robin constant of D with respect to x^o*. B. Robin [3] originally dealt with the case of $m=3$. When $m=2$, as is well known, the Robin constant plays an interesting role in the theory of Riemann surfaces.

Let $\boldsymbol{C}^n$ be the n-dimensional complex plane with the standard coordinate system $z=(z_1, \cdots, z_n)$, and Δ a unit disc with center at origin in the 1-dimensional complex plane $\boldsymbol{C}$. Consider a domain $\mathcal{D}$ over $\Delta\times \boldsymbol{C}^n$, precisely speaking, $\mathcal{D}$ is an unramified covering domain over the product space $\Delta\times \boldsymbol{C}^n$ ($\subset \boldsymbol{C}^{n+1}$). We set $\mathcal{D}(t)=\mathcal{D}\cap(\{t\}\times \boldsymbol{C}^n)$ for $t\in\Delta$, which is called the *fiber of $\mathcal{D}$ at $t\in\Delta$*. We regard the domain $\mathcal{D}$ of dimension $n+1$ as a variation of domains $\mathcal{D}(t)$ of dimension n with parameter $t\in\Delta$, and write it $\mathcal{D}: t\to\mathcal{D}(t)$ where $t\in\Delta$. Let α be a *holomorphic section of* $\mathcal{D}$ *on* Δ, that is, α is a holomorphic map of Δ into $\mathcal{D}$ such that $\alpha(t)\in\mathcal{D}(t)$ for all $t\in\Delta$. Putting $z_i=x_{2i-1}+\sqrt{-1}\,x_{2i}$ $(i=1,\cdots,n)$ with x_{2i-1} and x_{2i} real, we consider $\mathcal{D}(t)$ as a domain over the space $\boldsymbol{R}^{2n}$ with coordinate system $(x_1, x_2, \cdots, x_{2n-1}, x_{2n})$. Then we have the Green function $g(t,z)$ of $\mathcal{D}(t)$ with pole $\alpha(t)$ (by definition, we set $g(t,z)=0$ on each connected component of $\mathcal{D}(t)$ except for that containing $\alpha(t)$) and the Robin constant $\lambda(t)$ of $\mathcal{D}(t)$ with respect to $\alpha(t)$. Thus $\lambda(t)$ defines a real valued function on Δ. Our main result is the following

Theorem. *If* $\mathcal{D}$ *is a pseudoconvex domain of dimension* $n+1$, *then* $\lambda(t)$ *is superharmonic on* Δ. *Moreover,* $\log(-\lambda(t))$ *is subharmonic on* Δ *in the case of* $n\geqq 2$.

In the case of $n=1$, a proof of Theorem was given in [5] and in this case T. Nishino [1] made clear what amounts to. In the present note, we give a sketch of the proof of Theorem for $n\geqq 2$.

2. It suffices to consider the case in which all $\mathcal{D}(t)$ $(t\in\Delta)$ contain the origin $z=O$ of $\boldsymbol{C}^n$ and $\alpha(t)=O$ on all $t\in\Delta$.

Step 1. Suppose that there exists another domain $\tilde{\mathcal{D}}$ over $\Delta\times\boldsymbol{C}^n$ and a real valued analytic function ψ defined on $\tilde{\mathcal{D}}$ such that (i) ψ is plurisubharmonic on $\tilde{\mathcal{D}}$, (ii) $\mathcal{D}\subset\tilde{\mathcal{D}}$ and $\mathcal{D}(t)$ are relatively compact in $\tilde{\mathcal{D}}(t)$ for all $t\in\Delta$, (iii) $\mathcal{D}=\{(t,z)\in\tilde{\mathcal{D}}\mid\psi(t,z)<0\}$ and

$$\partial\mathcal{D}=\{(t,z)\in\tilde{\mathcal{D}}\mid\psi(t,z)=0\},$$

where $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$ in $\tilde{\mathcal{D}}$, (iv) each $\partial\mathcal{D}(t)$ $(t\in\Delta)$ are non-singular, that is, $(\partial\psi/\partial z_i)_{1\leq i\leqq n}\neq O$ on $\partial\mathcal{D}(t)$. The last condition (iv) implies that the variation $\mathcal{D}: t\to\mathcal{D}(t)$ where $t\in\Delta$ is diffeomorphically trivial, and that $g(t,z)$ (resp. $\lambda(t)$) is of class C^2 on $\mathcal{D}\cup\partial\mathcal{D}$ (resp. Δ). Then we obtain the following

Lemma. *We have the inequality*

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}\leqq-\frac{2\Gamma(n-1)}{\pi^n}\iint_{\mathcal{D}(t)}\left\{\sum_{i=1}^{n}\left|\frac{\partial^2 g(t,z)}{\partial t\partial\bar{z}_i}\right|^2\right\}dV,$$

where $dV=dx_1dx_2\cdots dx_{2n-1}dx_{2n}$ *is the volume element of* $\boldsymbol{R}^{2n}$.

It follows from Lemma that $\lambda(t)$ is superharmonic on Δ, provided that the above conditions (i)–(iv) are satisfied.

Step 2. We suppose that $\mathcal{D}$ satisfies the above conditions (i), (ii), (iii) except for (iv). Then we do not know if $\lambda(t)$ is of class C^2 on Δ. However, using the fact that $g(t,z)$ is continuous on $\mathcal{D}\cup\partial\mathcal{D}$ and for any fixed $t\in\Delta$, the function $\psi(t,z)-g(t,z)$ is subharmonic on $\mathcal{D}(t)$, we find that $\lambda(t)$ *is of class* C^1 *on* Δ. Since the set of points t of Δ such that $\partial\mathcal{D}(t)$ fails to satisfy the condition (iv), consists of real 1-dimensional curves, we infer from Step 1 that $\lambda(t)$ is superharmonic on Δ.

Step 3. Suppose that $\mathcal{D}$ satisfies the conditions (i), (ii) and (iii). Let $\varphi(t)$ be an arbitrary holomorphic function on Δ such that $\varphi(t)\neq 0$

at any $t \in \Delta$. We consider the Hartogs transformation T of the form, $(t, z) \mapsto (t, Z) = (t, \sqrt[2n-2]{\varphi(t)}z)$. Set $\mathcal{D}^* = T(\mathcal{D})$ and $\mathcal{D}^*(t) = T(\mathcal{D}(t))$ for $t \in \Delta$, and let $g^*(t, Z)$ and $\lambda^*(t)$ denote respectively the Green function on $\mathcal{D}^*(t)$ with pole O and the Robin constant of $\mathcal{D}^*(t)$ with respect to O. Then we get $g^*(t, Z) = g(t, z)/|\varphi(t)|$ and $\lambda^*(t) = \lambda(t)/|\varphi(t)|$. Since $\mathcal{D}^*$ satisfies the conditions (i), (ii) and (iii), it follows from Step 2 that $\lambda^*(t)$ is superharmonic on Δ. Consequently, $\log(-\lambda(t))$ is subharmonic on Δ.

Step 4. Let $\mathcal{D}$ be a general pseudoconvex domain over $\Delta \times \boldsymbol{C}^n$. By Oka's theorem ([2], p. 143), there exists a sequence of subdomains $\mathcal{D}_p$ $(p = 1, 2, \cdots)$ of $\mathcal{D}$ such that $\Delta_p \subset \Delta_{p+1}$, $\bigcup_{p=1}^{\infty} \Delta_p = \Delta$, $\mathcal{D}_p \subset \mathcal{D}_{p+1}$, $\bigcup_{p=1}^{\infty} \mathcal{D}_p = \mathcal{D}$, and that each $\mathcal{D}_p$ is a domain over $\Delta_p \times \boldsymbol{C}^n$ which satisfies the conditions (i), (ii) and (iii). Denoting by $\lambda_p(t)$ the Robin constant of $\mathcal{D}_p(t)$ with respect to O, we have that $\lambda_p(t) \leqq \lambda_{p+1}(t)$ and $\lim_{p \to \infty} \lambda_p(t) = \lambda(t)$ for $t \in \Delta$. It follows from Step 3 that $\log(-\lambda(t))$ is subharmonic on Δ. Thus the proof is completed.

3. We give some applications of Theorem for $n \geqq 2$ and compare them with those for $n = 1$.

(a) (Fiber uniformity). A domain D over $\boldsymbol{C}^n$ $(n \geqq 1)$ is said to be parabolic, if the Robin constant λ of D with respect to some (hence any) point z^o of D is $+\infty$ (resp. $=0$) for $n=1$ (resp. $n \geqq 2$). *Let $\mathcal{D}$ be a pseudoconvex domain over $\Delta \times \boldsymbol{C}^n$ and set $K = \{t \in \Delta \mid \mathcal{D}(t)$ is parabolic$\}$. Then, if the logarithmic capacity of K on the complex plane $\boldsymbol{C}$ is positive, we have $K = \Delta$.*

(b) (Trivial variations). *Let $\mathcal{D}$ be a pseudoconvex domain over $\Delta \times \boldsymbol{C}^n$ $(n \geqq 1)$. In the case of $n \geqq 2$, if there exists a holomorphic section α of $\mathcal{D}$ on Δ such that $\lambda(t)$ is harmonic on Δ, then $\mathcal{D}$ is identical with the trivial variation: $t \to \mathcal{D}(0) + \alpha(t)$ where $t \in \Delta$.* Let $n = 1$ and χ denote the Euler characteristic number of $\mathcal{D}(0)$. If there exist at least $\chi + 1$ holomorphic sections α_i $(i = 1, \cdots, \chi+1)$ of $\mathcal{D}$ on Δ such that each $\lambda_i(t)$ is harmonic on Δ, then $\mathcal{D}$ is holomorphically isomorphic to the trivial variation: $t \to \mathcal{D}(0)$ where $t \in \Delta$ ([6], p. 344).

(c) (Metric induced by the Robin constant). Let D be a domain over $\boldsymbol{C}^n$ $(n \geqq 2)$ with non-singular analytic boundary. For a point $z \in D$, we denote by $\lambda(z)$ the Robin constant of D with respect to z. Then $\lambda(z)$ defines a real negatively valued function on D such that $\lambda(z) \cdot d(z, \partial D)^{2n-2}$ is bounded for $z \in D$ near ∂D, where $d(z, \partial D)$ is the euclidean distance from z to ∂D. We infer from Lemma that, *if D is pseudoconvex, then $\log(-\lambda(z))$ is strongly plurisubharmonic on D. Thus, $ds^2 = \sum_{i,j=1}^{n} (\partial^2 \log(-\lambda(z)/\partial z_i \partial \bar{z}_j) dz_i d\bar{z}_j$ defines a complete metric on D.* In the case of $n = 1$, N. Suita [4] showed that it is identical, apart from a constant factor, with the Bergman metric on any hyperbolic Riemann surface.

4. We study variations of domains in $\boldsymbol{R}^m$ with $m\geqq 3$. Let I be an open interval of the real line $\boldsymbol{R}$. Consider a univalent domain $\mathcal{D}$ of the product space $I\times\boldsymbol{R}^m$ ($\subset\boldsymbol{R}^{m+1}$) and set $\mathcal{D}(t)=\mathcal{D}\cap(\{t\}\times\boldsymbol{R}^m)$ for $t\in I$. Let α be a section of $\mathcal{D}$ on I of the form $\alpha(t)=at+b$ for $t\in I$, where $a, b\in\boldsymbol{R}^m$. For each $t\in I$, we denote respectively by $g(t, x)$ and $\lambda(t)$ the Green function on $\mathcal{D}(t)$ with pole $\alpha(t)$ and the Robin constant of $\mathcal{D}(t)$ with respect to $\alpha(t)$. Then, by similar arguments to those of §§ 2 and 3, we have the following results:

(1) *If $\mathcal{D}$ is a convex domain of $\boldsymbol{R}^{m+1}$ with real analytic boundary in $I\times\boldsymbol{R}^m$, then we have the inequality*

$$\frac{\partial^2\lambda(t)}{\partial t^2}\leqq-\frac{\Gamma(m/2-1)}{2\pi^{m/2}}\iint_{\mathcal{D}(t)}\left\{\sum_{i=1}^{m}\left(\frac{\partial^2 g(t, x)}{\partial t\partial x_i}\right)^2\right\}dV,$$

where $dV=dx_1\cdots dx_m$ denotes the volume element of $\boldsymbol{R}^m$. Moreover, $\log(-\lambda(t))$ *is a convex function on I.*

(2) *Let D be a convex domain in $\boldsymbol{R}^m$ with real analytic boundary. Let $\lambda(x)$ denote the Robin constant of D with respect to $x\in D$. Then $ds^2=\sum_{i,j=1}^m \partial^2 \log(-\lambda(x)/\partial x_i\partial x_j)dx_i dx_j$ defines a complete metric on D.*

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