74. *Asymptotic Error Estimation for Spline-on-Spline Interpolation*

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1. Introduction and description of method. We shall consider an asymptotic error estimation for spline-on-spline interpolation on a uniform mesh. The spline-on-spline technique is described in [3]: Suppose we have an interval [0, 1] partitioned in n equal parts of length h. In order to find for a given function $f(x)$ an approximant of its first derivative $f'(x)$, we interpolate $f(x)$ by a cubic spline $s(x)$ over the mesh under appropriate additional end conditions at the endpoints. Then replacing $f(x)$ by $s'(x)$, we get an approximant of $f''(x)$, and this procedure can be continued. Ahlberg *et al.* [1] observed that it gives excellent results for the second derivative of sin x, and Dolezal and Tewarson [3] obtained error bounds for the spline-on-spline interpolation. In what follows, let r, k and l be nonnegative integers, and $s_i^{(j)}=s^{(j)}(ih)$.

Corresponding to the sufficiently smooth function $f(x)$ on [0, 1], one constructs a cubic spline $s(x)$ of the form

$$s(x)=\sum_{i=-3}^{n-1}\alpha_i Q_4(x/h-i), \qquad nh=1 \tag{1}$$

so that

$$s_i=f_i, \qquad i=0, 1, \cdots, n. \tag{2}$$

Since s depends upon $n+3$ parameters, two additional conditions are required toward the determination of s. Under various (end) conditions in [5], we have

$$f'_i-s'_i=(h^4/180)f_i^{(5)}+O(h^5), \qquad i=0, 1, \cdots, n. \tag{3}$$

In the present paper we take these to be homogeneous end conditions:

$$\Delta^r s'_0=\nabla^r s'_n=0 \tag{4}$$

where Δ and ∇ are forward and backward difference operators, respectively. By the consistency relation [1, p. 13] and (2), we have

$$\begin{aligned}(1/6)(s'_{i+1}+4s'_i+s'_{i-1})&=(1/2)h^{-1}(s_{i+1}-s_{i-1})\\&=(1/2)h^{-1}(f_{i+1}-f_{i-1}), \quad i=1, 2, \cdots, n-1.\end{aligned} \tag{5}$$

By (5), end condition $\Delta^r s'_0=0$ may be equivalently rewritten as follows

$$s'_0+a_r s'_1=L_r(f_0, f_1, \cdots, f_r), \qquad r\neq 2 \tag{6}$$

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where a_r is real constant and $L_r(f_0, f_1, \cdots, f_r)$ is some linear combination of f_i, $i=0, 1, \cdots, r$. For example, $\Delta^6 s_0'=0$ is equivalent to

$$s_0'+(15/4)s_1'=(1/144)(865d_1-226d_2+54d_3-10d_4+d_5) \tag{7}$$

where we denote the right-hand side of (5) by d_i. For $\{a_r\}$, we have

Lemma 1. *The above sequence $\{a_r\}$ is defined in the following manner:*

$$(a_0, a_1, a_3)=(0, -1, 5)$$
$$a_{r+1}=(5a_r-1)/(a_r+1), \qquad r=3, 4, \cdots.$$

In addition

$$\lim_{r\to\infty} a_r=2+\sqrt{3}.$$

Proof. Since $\Delta^{r+1}s_0'=\Delta^r s_1'-\Delta^r s_0'$, by mathematical induction and consistency relation (5), we have the desired result.

Before we proceed with analysis, we shall require the following lemma which is similarly obtained in the proof of Lemma in [7].

Lemma 2 (cf. [2]). *If $(2+\sqrt{3})\alpha \neq \beta$, the following tridiagonal n by n matrix A_n is nonsingular for sufficiently large n and in addition*

$$\|A_n^{-1}\| \leq C \text{ for some constant } C \text{ independent of } n \tag{8}$$

where $\|\cdot\|$ stands for the matrix maximum norm and

$$A_n=\begin{bmatrix} \alpha & \beta & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & \beta & \alpha \end{bmatrix}.$$

With the help of Lemmas 1 and 2, the resulting matrix generated by (2) and (4) will be ill-conditioned for large integers r.

From (4) and (5), by Taylor series expansion we have

$$\begin{cases} \Delta^r e_0'=O(h^r) \\ (1/6)(e_{i+1}'+4e_i'+e_{i-1}')=(h^4/180)f_i^{(5)}+(h^6/3780)f_i^{(7)}+O(h^8), \\ \qquad\qquad\qquad\qquad i=1, 2, \cdots, n-1 \\ \nabla^r e_n'=O(h^r) \end{cases} \tag{9}$$

where $e=f-s$.

Since a_r is rational, in virtue of Lemma 2 we have

$$f_i'-s_i'=(h^4/180)f_i^{(5)}-(h^6/1512)f_i^{(7)}+O(h^{\min(8,r)}) \tag{10}$$
$$i=0, 1, \cdots, n \text{ for sufficiently small } h.$$

Now by means of spline-on-spline technique, let us consider a cubic spline p of the form

$$p(x)=\sum_{i=-3}^{n-1} \beta_i Q_4(x/h-i) \tag{11}$$

so that

$$\begin{cases} \Delta^k p_0'=0, & k\geq r-1 \\ p_i=s_i', & i=0, 1, \cdots, n \\ \nabla^k p_n'=0. \end{cases} \tag{12}$$

That is, p is considered to be a cubic spline-on-spline interpolant of s'. By the consistency relation and (11), we have

(13) $$(1/6)(p'_{i+1}+4p'_i+p'_{i-1})=(1/2)h^{-1}(s'_{i+1}-s'_{i-1})$$
$$=f''_i+(h^2/6)f_i^{(4)}+(h^4/360)f_i^{(6)}-(h^6/15120)f_i^{(8)}+O(h^{\min(7,r-1)}),$$
$$i=1, 2, \cdots, n-1.$$

Hence, by using again Lemma 2, we get

(14) $$f''_i-p'_i=(h^4/90)f_i^{(6)}-(h^6/756)f_i^{(8)}+O(h^{\min(7,r-1)}), \quad i=0, 1, \cdots, n.$$

Since $f''_i-s''_i=(h^2/12)f_i^{(4)}+\cdots$ ([5]), by (14) we have an improved approximation to f'' at mesh point. Here we remark that the resulting matrix generated by (2) and (4) is exactly the same one generated by (12) for $k=r$, i.e., the spline-on-spline p is determined with very little additional effort.

Here we consider a cubic spline q of the form

$$q(x)=\sum_{i=-3}^{n-1} \gamma_i Q_4(x/h-i)$$

so that

(15) $$\begin{cases} \Delta^l q'_0=0, & l\geq r-2 \\ q_i=p'_i, & i=0, 1, \cdots, n \\ \nabla^l q'_n=0. \end{cases}$$

That is, q is considered to be a spline-on-spline interpolant of p'. Similarly we have

(16) $$f_i^{(3)}-q'_i=(h^4/60)f_i^{(7)}=O(h^{\min(6,r-2)}) \qquad i=0, 1, \cdots, n.$$

By means of Kershaw's technique in [4], we have

Remark. For any nonnegative integers k, r and l, we have (14) and (16) for any mesh point different from endpoints $x=0, 1$.

2. Results of some numerical experiments. The results of some computational experiments are given in Table for the functions e^x and e^{5x}. We choose $(r, n)=(6, 32)$ and denote

$$k_2(k)=\{f''(x)-p'(x)\}/\{(h^4/90)f^{(6)}(x)\}$$
$$k_3(x)=\{f^{(3)}(x)-q'(x)\}/\{(h^4/60)f^{(7)}(x)\}.$$

By (14) and (16), values of $k_2(x)$ and $k_3(x)$ converge to 1 for any fixed mesh points $x\in(0, 1)$ as $h\to 0$.

Table

x	e^x		e^{5x}	
	$k_2(x)$	$k_3(x)$	$k_2(x)$	$k_3(x)$
1/4	1.00	1.05	1.00	1.03
1/2	1.00	1.00	1.00	1.00
3/4	1.00	1.07	0.99	1.13

References

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