63. Toda Lattice Hierarchy. II

By Kimio UENO*) and Kanehisa TAKASAKI**)

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0. Introduction. This note is a sequel to the preceding paper [1]. Here we shall discuss the l-reduced family, $(\mathrm{TL})_l$, of the Toda lattice (TL) hierarchy, and shall construct the N-soliton solutions by making use of the Riemann-Hilbert decomposition, which is an infinite dimensional generalization of the classical Riemann-Hilbert problem.

The TL hierarchy was introduced in [1] as follows: Let $x=(x_1, x_2, \cdots)$, $y=(y_1, y_2, \cdots)$ be two time flows. Let L, M, B_n, C_n $(n=1, 2, \cdots)$ be matrices of size $Z\times Z$ of the form

$$(1)\qquad \begin{aligned} L&=\sum_{-\infty<j\leq 1}\operatorname{diag}[b_j(s;x,y)]\Lambda^j, \qquad && b_1(s;x,y)=1,\\ M&=\sum_{-1\leq j<+\infty}\operatorname{diag}[c_j(s;x,y)]\Lambda^j, \qquad && c_{-1}(s;x,y)\neq 0,\end{aligned}$$

$$(2)\qquad B_n=(L^n)_+, \qquad C_n=(M^n)_-.$$

(As for the notations, the readers should refer to [1].) Then the TL hierarchy is defined by the following Zakharov-Shabat equations;

$$(3)\qquad \begin{aligned} &\partial_{x_n}B_m-\partial_{x_m}B_n+[B_m, B_n]=0, \qquad && \partial_{y_n}C_m-\partial_{y_m}C_n+[C_m, C_n]=0,\\ &\partial_{y_n}B_m-\partial_{x_m}C_n+[B_m, C_n]=0, \qquad && m, n=1, 2, \cdots.\end{aligned}$$

The hierarchy is linearized by the equations

$$(4)\qquad \begin{aligned} &LW^{(\infty)}=W^{(\infty)}\Lambda, \qquad MW^{(0)}=W^{(0)}\Lambda^{-1},\\ &\partial_{x_n}W=B_nW, \quad \partial_{y_n}W=C_nW, \quad W=W^{(\infty)}, W^{(0)}, \quad n=1, 2, \cdots.\end{aligned}$$

As fundamental solution matrices to these equations, one may choose matrices of the form

$$(5)\qquad \begin{aligned} &W^{(\infty)}(x,y)=\hat{W}^{(\infty)}(x,y)\exp\xi(x,\Lambda),\\ &W^{(0)}(x,y)=\hat{W}^{(0)}(x,y)\exp\xi(y,\Lambda^{-1}), \qquad \xi(x,\Lambda^{\pm 1})=\sum_{n=1}^{\infty}x_n\Lambda^{\pm n},\\ &\hat{W}^{\binom{0}{\infty}}(x,y)=\sum_{j=0}^{\infty}\operatorname{diag}[\hat{w}_j^{\binom{0}{\infty}}(s;x,y)]\Lambda^{\pm j}.\end{aligned}$$

Such fundamental solution matrices will be referred to as wave matrices.

The linearization (4) equivalently leads to the bilinear relation

$$(6)\qquad W^{(\infty)}(x,y)W^{(\infty)}(x',y')^{-1}=W^{(0)}(x,y)W^{(0)}(x',y')^{-1} \quad \text{for any } x, x', y, y'.$$

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*) Department of Mathematics, Yokohama City University, 22-2 Seto, Yokohama 236.

**) Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo 113.

From this relation, we deduce the existence of τ functions, $\tau(s;x,y)$, defined by

$$(7)\qquad \begin{aligned}\hat{w}_j^{(\infty)}(s;x,y)&=p_j(-\tilde{\partial}_x)\tau(s;x,y)/\tau(s;x,y),\\ \hat{w}_j^{(0)}(s;x,y)&=p_j(-\tilde{\partial}_y)\tau(s+1;x,y)/\tau(s;x,y)\end{aligned}$$

where $p_j(x)$ is introduced through $\sum_{j=0}^{\infty} p_j(x)\lambda^j=\exp\xi(x,\lambda)$. Substituting (7) to (6), the TL hierarchy is converted to Hirota's bilinear equations.

1. l-periodic reduction. Let l be a positive integer. Let us consider a subfamily of the TL hierarchy with the additional constraints

$$(8)\qquad L^l=\Lambda^l,\qquad M^{-l}=\Lambda^{-l}.$$

This subfamily will be referred to as the l-periodic Toda lattice ($(\mathrm{TL})_l$) hierarchy. One easily sees that the l-periodic condition (8) yields

$$(9)\qquad \begin{aligned}&[L,\Lambda^n]=[M,\Lambda^n]=0,\qquad [W^{(\infty)},\Lambda^n]=[W^{(0)},\Lambda^n]=0,\\ &\partial_{x_n}L=\partial_{y_n}L=0,\quad \partial_{x_n}M=\partial_{y_n}M=0\qquad \text{for } n\equiv 0 \bmod l.\end{aligned}$$

That is, solutions of the $(\mathrm{TL})_l$ hierarchy are independent of x_n, y_n ($n\equiv 0 \bmod l$). Let us introduce τ functions, $\tau'(s;x,y)$, by $\tau'(s;x,y)=\exp\left(\sum_{n=1}^{\infty} nx_ny_n\right)\tau(s;x,y)$ (cf. [1]). Then we obtain

Proposition 1. *Under a suitable choice of elementary multipliers of τ functions* (cf. [1], Theorem 4), *one finds that the l-periodic condition* (8) *equivalently reads*

$$(10)\qquad \begin{aligned}&\partial_{x_n}\tau'(s;x,y)=\partial_{y_n}\tau'(s;x,y)=0,\\ &\tau'(s;x,y)=\tau'(s+l;x,y),\qquad \textit{for any } s,\ n\equiv 0 \bmod l.\end{aligned}$$

In order to investigate the linearization scheme for the $(\mathrm{TL})_l$ hierarchy (cf. [4]), let us recall some basic facts about Lie algebras. Let $\mathfrak{gl}((\infty))$ be the formal Lie algebra of matrices of size $\boldsymbol{Z}\times\boldsymbol{Z}$, and $\mathfrak{gl}((\infty))_l$ be a subalgebra of $\mathfrak{gl}((\infty))_l$ defined by

$$\mathfrak{gl}((\infty))_l=\{A\in\mathfrak{gl}((\infty))\,|\,[A,\Lambda^n]=0\qquad \text{for } n\equiv 0 \bmod l\}$$

$\mathfrak{gl}(l,\boldsymbol{C}[[\zeta,\zeta^{-1}]])$ is isomorphic to $\mathfrak{gl}((\infty))_l$ under the correspondence

$$(11)\qquad \begin{aligned}&\mathfrak{gl}(l,\boldsymbol{C}[[\zeta,\zeta^{-1}]])\longrightarrow\mathfrak{gl}((\infty))_l\\ &A(\zeta)=\sum_{j\in\boldsymbol{Z}}\mathrm{diag}[a_j(0),\cdots,a_j(l-1)]\Lambda_l(\zeta)^j\\ &\qquad\longmapsto A=\sum_{j\in\boldsymbol{Z}}\mathrm{diag}[a_j(0),\cdots,a_j(l-1)]_l\Lambda^j,\end{aligned}$$

where

$$\Lambda_l(\zeta)=\begin{bmatrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\\zeta&&&0\end{bmatrix},$$

and $\mathrm{diag}[a_j(0),\cdots,a_j(l-1)]_l$ denotes a diagonal matrix $\mathrm{diag}(\cdots,a_j(0),\cdots,a_j(l-1),a_j(0),\cdots,a_j(l-1),\cdots]$. Under this isomorphism, L, M, $W^{(\infty)}(x,y)$, $W^{(0)}(x,y)$ are identified with $L(\zeta)$, $M(\zeta)$, $W^{(\infty)}(x,y;\zeta)$, $W^{(0)}(x,y;\zeta)\in\mathfrak{gl}(l,\boldsymbol{C}[[\zeta,\zeta^{-1}]])$, which take the form

$$L(\zeta)=W^{(\infty)}(x, y ; \zeta)\Lambda_l(\zeta)W^{(\infty)}(x, y ; \zeta)^{-1},$$
$$M(\zeta)=W^{(0)}(x, y ; \zeta)\Lambda_l(\zeta)^{-1}W^{(0)}(x, y ; \zeta)^{-1},$$
$$W^{(\infty)}(x, y ; \zeta)=\hat{W}^{(\infty)}(x, y ; \zeta)\exp \xi(x, \Lambda_l(\zeta)),$$
$$W^{(0)}(x, y ; \zeta)=\hat{W}^{(0)}(x, y ; \zeta)\exp \xi(y, \Lambda_l(\zeta)^{-1}),$$
$$\hat{W}^{\binom{0}{\infty}}(x, y ; \zeta)=\sum_{j=0}^{\infty}\mathrm{diag}[\hat{w}_j^{\binom{0}{\infty}}(0 ; x, y), \cdots, \hat{w}_j^{\binom{0}{\infty}}(l-1 ; x, y)]\Lambda_l(\zeta)^{\pm j}.$$

Proposition 2. *$W^{\binom{0}{\infty}}(x, y ; \zeta)$ solve the linear equations*

(12) $\quad \partial_{x_n}W(\zeta)=B_n(\zeta)W(\zeta), \quad \partial_{y_n}W(\zeta)=C_n(\zeta)W(\zeta), \quad n=1, 2, \cdots,$

where $B_n(\zeta)=[L(\zeta)^n]_+$, $C_n(\zeta)=[M(\zeta)^n]_-$. The symbols $[A(\zeta)]_\pm$ stand for the part of $A(\zeta)$ of non-negative order, and of strictly negative order with respect to $\Lambda_l(\zeta)$, respectively. The compatibility condition for (12) *gives the (TL) hierarchy, and $B_n(\zeta)$, $C_n(\zeta)$ belong to $\mathfrak{gl}(l, \boldsymbol{C}[\zeta, \zeta^{-1}])$.*

2. The Riemann-Hilbert decomposition and its application. We define $\hat{V}^{\binom{0}{\infty}}(x, y)$ for wave matrices $W^{\binom{0}{\infty}}(x, y)$ by $W^{\binom{0}{\infty}}(x, y)=\hat{V}^{\binom{0}{\infty}}(x, y)\exp(\xi(x, \Lambda)+\xi(y, \Lambda^{-1}))$. They take the form $\hat{V}^{\binom{0}{\infty}}(x, y)=\sum_{j=0}^{\infty}\mathrm{diag}[\hat{v}_j^{\binom{0}{\infty}}(s : x, y)]\Lambda^{\pm j}$, and $\hat{v}_0^{(\infty)}(s ; x, y)=1$. Then the bilinear relation (6) implies that there exists an invertible matrix A of size $\boldsymbol{Z}\times \boldsymbol{Z}$ such that

(13) $$\hat{V}^{(0)}(x, y)=\hat{V}^{(\infty)}(x, y)H(x, y)$$

where $H(x, y)=\exp(\xi(x, \Lambda)+\xi(y, \Lambda^{-1}))\ A\ \exp(\xi(-x, \Lambda)+\xi(-y, \Lambda^{-1}))$. This implies decomposition of $H(x, y)$ to the upper triangular matrix $\hat{V}^{(0)}$ and the lower triangular matrix $\hat{V}^{(\infty)}$, and is regarded as an infinite dimensional generalization of the classical Riemann-Hilbert (RH) problem. In fact, if $A\in \mathfrak{gl}((\infty))_l$ then (13) reduces to the classical RH problem. Therefore (13) will be called the RH decomposition.

Proposition 3. *Let $A=I+\sum_{j=1}^{N}a_jX_{p_jq_j}$ in* (13), *where*

$$X_{pq}=\sum_{m,n\in \boldsymbol{Z}}p^mq^{-n}E_{mn}\ (E_{mn}=(\delta_{mi}\delta_{nj})_{i,j\in \boldsymbol{Z}})$$

is the so-called vertex operator [3]. *Suppose $a_j>0$, and $0<q_N<\cdots<q_1<p_1<\cdots<p_N$. Then the RH decomposition has a unique pair of solutions $\hat{V}^{\binom{0}{\infty}}(x, y)$, and their entries are expressed in the form*

$$\hat{v}_j^{(\infty)}(s ; x, y)=p_j(-\tilde{\partial}_x)\tau'(s ; x, y)/\tau'(s ; x, y),$$
$$\hat{v}_j^{(0)}(s ; x, y)=p_j(-\tilde{\partial}_y)\tau'(s+1 ; x, y)/\tau'(s ; x, y).$$

Here the τ functions, $\tau'(s ; x, y)$, is given by

(14) $$\tau'(s ; x, y)=\sum_{l=0}^{N}\sum_{i_1<\cdots<i_l}c_{i_1\cdots i_l}a_{i_1}(s)\cdots a_{i_l}(s)\exp\left(\sum_{\mu=1}^{l}\eta(p_{i_\mu})-\eta(q_{i_\mu})\right)$$

where

$$a_i(s)=a_i(p_i/q_i)^sq_i/(p_i-q_i),$$
$$c_{i_1\cdots i_l}=\sum_{1\leqq\mu<\nu\leqq i}c_{i_\mu i_\nu}, \quad c_{ij}=(p_i-p_j)(q_i-q_j)/(p_i-q_j)(q_i-p_j),$$
$$\eta(p)=\xi(x, p)+\xi(y, p^{-1}).$$

The τ function (14) coincides with the τ function introduced in [5], [6].

3. Remarks. (1) Setting $A=I+\sum_{j=1}^{r} a_j E_{m_j n_j}$ and $y=0$ in (13), one can obtain rational solutions of the KP hierarchy (cf. Theorem 5 in [1]).

(2) The TL hierarchies of orthogonal or symplectic type, and the multi-component TL hierarchy can be also considered.

These topics are investigated in detail in [2].

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