46. A Generalization of Gauss' Theorem on Arithmetic-Geometric Means

By Takashi ONO
Department of Mathematics, Johns Hopkins University

(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1983)

§ 1. **Introduction and methods.** With each continuous map f: $\boldsymbol{R}^n \to \boldsymbol{R}^m$ we associate an entire function $f^\#(z)$ given by

$$f^\#(z)=\int_{S^{n-1}} e^{zN(f(x))}d\omega_{n-1}.^{*)}$$

We shall assume throughout that

(1.1) $$f(x)\neq 0 \quad \text{for all } x\in S^{n-1},$$

hence $N(f(x))>0$ on S^{n-1}. When it is so, the integral

(1.2) $$\Gamma(f\,;s)=\int_0^\infty t^{s-1}f^\#(-t)dt$$

represents a holomorphic function for $\sigma=\operatorname{Re} s>0$. We have

(1.3) $$\Gamma(f\,;s)=\Gamma(s)K(f\,;s)$$

where $\Gamma(s)$ is the usual gamma function and

(1.4) $$K(f\,;s)=\int_{S^{n-1}} N(f(x))^{-s}d\omega_{n-1}.$$

By (1.1), $K(f\,;s)$ is entire and (1.3) yields the meromorphic continuation of $\Gamma(f\,;s)$ onto $\boldsymbol{C}$.

When $n=m=2$, $f(x)=(ax_1, bx_2)$, $0<a\leqq b$ and $s=1/2$, our $K(f\,;s)$ becomes the complete elliptic integral:

$$K\left(f\,;\frac{1}{2}\right)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}.$$

Gauss proved, by means of quadratic transformations of theta series,

(G) $$K\left(f\,;\frac{1}{2}\right)=K\left(f_1\,;\frac{1}{2}\right), \qquad f_1(x)=(a_1x_1, b_1x_2)$$

where $a_1=\sqrt{ab}$, $b_1=(a+b)/2$.**) The repeated application of (G) yields immediately the relation $K(f\,;1/2)=M(a,b)^{-1}$ where $M(a,b)$ means the arithmetic-geometric of a, b.

In this paper, we shall generalize (G) for our $K(f\,;s)$ defined by (1.4) when $n=m=2p$, $p>\sigma=\operatorname{Re} s>(p-1)/2$ and $f(x)=(ax_1,\cdots,ax_p, bx_{p+1},\cdots,bx_{2p})$. The proof depends on the fact that, under the assumptions, $K(f\,;s)$ can be expressed as a hypergeometric series via

*) We denote by $\langle x, y\rangle$ the standard inner product in $\boldsymbol{R}^n$. We put $Nx=\langle x,x\rangle$. The unit sphere is $S^{n-1}=\{x\in \boldsymbol{R}^n;\, Nx=1\}$. We denote by $d\omega_{n-1}$ the volume element of S^{n-1} such that the volume of S^{n-1} is 1.

**) See [1] p. 352. See also [7] § 9 and [8] p. 269.

Gegenbauer polynomials. The quadratic transformation of hypergeometric series takes the place of that of theta series in Gauss' case.

Back to the general situation, the Taylor expansion at 0 of the entire function $f^{\#}(z)$ is given by

$$(1.5)\qquad f^{\#}(z)=\sum_{k=0}^{\infty} K(f\,;\,-k)\frac{z^k}{k!}.$$

Therefore, by (1.2), (1.3), (1.5), we have

$$(1.6)\qquad \Gamma(f\,;s)=\Gamma(s)K(f\,;s)=\int_0^{\infty} t^{s-1}\,dt\sum_{k=0}^{\infty} K(f\,;\,-k)\frac{(-t)^k}{k!}.\ ^{*)}$$

§ 2. **Linear maps.** When $f: \boldsymbol{R}^n\to\boldsymbol{R}^m$ is a linear map satisfying (1.1), calling $\lambda=(\lambda_1,\cdots,\lambda_n)$ the arbitrarily ordered set of eigenvalues of the quadratic form $N(f(x))$, we have

$$K(f\,;\,-k)=\frac{b_k(2\,;\lambda)}{b_k(2\,;1_n)},\qquad 1_n=(1,\cdots,1)\in\boldsymbol{R}^n$$

where the numbers $b_k(2\,;\lambda)$ are defined by the generating relation

$$(2.1)\qquad \sum_{k=0}^{\infty} b_k(2\,;\lambda)t^k=\prod_{i=1}^{n}(1-4\lambda_i t)^{-1/2}.\ ^{**)}$$

In particular, we have

$$b_k(2\,;1_n)=\frac{4^k(n/2,k)}{k!}.\ ^{***)}$$

Therefore, (1.6) becomes

$$(2.2)\qquad \Gamma(f\,;s)=\int_0^{\infty} t^{s-1}dt\sum_{k=0}^{\infty}\frac{b_k(2\,;\lambda)}{(n/2,k)}\left(\frac{-t}{4}\right)^k.$$

§ 3. **Certain diagonal maps.** Assume now that $n=m=2p$ and consider the following diagonal map

$$f(x)=(ax_1,\cdots,ax_p,bx_{p+1},\cdots,bx_{2p}),\qquad 0<a\leqq b.$$

Clearly this map satisfies (1.1). In view of the generating relation of the Gegenbauer polynomials:

$$\sum_{k=0}^{\infty} C_k^{p/2}(x)z^k=(1-2xz+z^2)^{-p/2},$$

the relation (2.1) with $\lambda_1=\cdots=\lambda_p=a^2$, $\lambda_{p+1}=\cdots=\lambda_{2p}=b^2$ yields

$$b_k(2\,;\lambda)=C_k^{p/2}\left(\frac{a^2+b^2}{2ab}\right)(4ab)^k.$$

Hence (2.2) becomes

$$\Gamma(f\,;s)=\int_0^{\infty} t^{s-1}\,dt\sum_{k=0}^{\infty}\frac{C_k^{p/2}((a^2+b^2)/2ab)}{(p,k)}(-abt)^k$$

---

*) This shows that the values of $K(f;s)$ for $\sigma>0$ are determined by its values at 0 and negative integers $-k$. Compare Ramanujan's formula (B) on p. 186 of [2]. In [4], [5] we wrote $N_k(f)$ for $K(f,-k)$.

**) See § 1 of [4].

***) $(a,k)=\begin{cases}a(a+1)\cdots(a+k-1), & k\geqq 1\\ 1, & k=0\end{cases},\quad a\in\boldsymbol{C},\ k\in\boldsymbol{Z}.$

$$=\int_0^\infty t^{s-1}e^{-((a^2+b^2)/2)t}\,{}_0F_1\left(;\frac{p+1}{2};\left(\frac{(a^2-b^2)t}{4}\right)^2\right)dt^{*)}$$

$$=\Gamma(s)\left(\frac{a^2+b^2}{2}\right)^{-s}{}_2F_1\left(\frac{s}{2},\frac{s+1}{2};\frac{p+1}{2};\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\right)$$

$$=\Gamma(s)(ab)^{-s}{}_2F_1\left(s,p-s;\frac{p+1}{2};-\frac{(b-a)^2}{4ab}\right)$$

where the last equality is crucial and follows from a quadratic transformation formula of hypergeometric series.**) In other words, we have, by (1.3),

$$K(f;s)=(ab)^{-s}{}_2F_1\left(s,p-s;\frac{p+1}{2};-\frac{(b-a)^2}{4ab}\right). \tag{3.1}$$

In order to use the integral representation of hypergeometric series, assume that $p>\sigma>(p-1)/2$, $\sigma=\mathrm{Re}\,s$. Then, (3.1) becomes

$$K(f;s)=(ab)^{-s}\frac{\Gamma((p+1)/2)}{\Gamma(p-s)\Gamma(s-(p-1)/2)} \times\int_0^1 t^{p-s-1}(1-t)^{s-(p-1)/2-1}\left(1+\frac{(b-a)^2}{4ab}t\right)^{-s}dt. \tag{3.2}$$

If we put $t=\sin^2\theta$, then (3.2) becomes

$$K(f;s)=\frac{2\Gamma((p+1)/2)}{\Gamma(p-s)\Gamma(s-(p-1)/2)} \times\int_0^{\pi/2}(\sin\theta)^{2p-2s-1}(\cos\theta)^{2s-p}\left(ab+\frac{(b-a)^2}{4}\sin^2\theta\right)^{-s}d\theta.$$

Summarizing, we obtain

Theorem. *Let $0<a\leqq b$ and $a_1=\sqrt{ab}$, $b_1=(a+b)/2$. Assume that $p>\sigma>(p-1)/2$, $\sigma=\mathrm{Re}\,s$. Then, we have*

$$\int_{S^{2p-1}}\frac{d\omega_{2p-1}}{(a^2(x_1^2+\cdots+x_p^2)+b^2(x_{p+1}^2+\cdots+x_{2p}^2))^s} =\frac{2\Gamma((p+1)/2)}{\Gamma(p-s)\Gamma(s-(p-1)/2)}\int_0^{\pi/2}\frac{(\sin\theta)^{2p-2s-1}(\cos\theta)^{2s-p}}{(a_1^2\cos^2\theta+b_1^2\sin^2\theta)^s}d\theta.$$

References

[1] C. F. Gauss: Werke. vol. III, Göttingen (1876).

[2] G. H. Hardy: Ramanujan. Cambridge Univ. Press (1940).

[3] W. Magnus, F. Oberhettinger, and R. P. Soni: Formulas and Theorems for the Special Functions of Mathematical Physics. 3rd ed., Springer-Verlag, New York (1966).

*) This follows from the formula (7) on p. 278 of [6].

**) By this we mean the formula

(*) $x^{-2\alpha}{}_2F_1(\alpha,\alpha+1/2;\gamma;(x^2-1)/x^2)={}_2F_1(2\alpha,2\gamma-2\alpha-1;\gamma;(1-x)/2)$.

The last equality is the case $\alpha=s/2$, $\gamma=(p+1)/2$, $x=(a^2+b^2)/2ab$. The formula (*) is the same as the one on line 3 from the bottom of p. 50 of [3] via the variable change $z=(x^2-1)/x^2$.

[4] T. Ono: On a generalization of Laplace integrals (to appear in Nagoya Math. J.).
[5] ——: On deformations of Hopf maps and hypergeometric series (manuscript).
[6] E. D. Rainville: Special Functions. Macmillan, New York (1960).
[7] T. Takagi: Kinsei Sûgakushidan (in Japanese; Topics from the History of Mathematics of the 19th Century). Kawade, Tokyo (1943).
[8] J. Tannery et J. Molk: Eléments de la Théorie des Fonctions Elliptiques. vol. 4, Gauthiers-Villars, Paris (1896).