116. *A Stationary Free Boundary Problem for a Circular Flow with or without Surface Tension*[*)]

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§ 1. In this note we are concerned with a free boundary problem which is a model for a flow around a planet. The problem is stated as follows.

Problem. Given a unit circle Γ in $\boldsymbol{R}^2$, find a closed Jordan curve γ outside Γ and a function V such that

(1.1) $\Delta V=0 \quad \text{in } \Omega_\gamma$,

(1.2) $V|_\Gamma=0, \quad V|_\gamma=a$,

(1.3) $\frac{1}{2}|\nabla V|^2+Q+\sigma K_\gamma=\text{unknown constant on } \gamma$,

(1.4) $|\Omega_\gamma|=\omega_0$.

Here Ω_γ is a doubly connected domain between Γ and γ (see Fig. 1). Constants $a>0$, $\omega_0>0$ and $\sigma\geqq 0$ are given. σ is the surface tension coefficient. Q is a given function defined outside Γ. K_γ is the curvature of γ ($K_\gamma>0$ if γ is convex). $|\Omega_\gamma|$ denotes the area of Ω_γ.

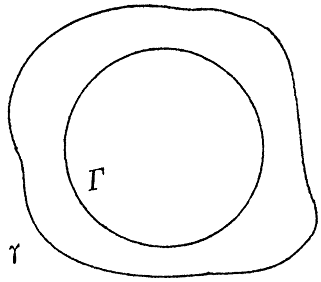

Fig. 1

Remark. We have assumed that the fluid is perfect, irrotational and that V is a stream function for the flow. Ω_γ is the flow region.

The more precise physical meaning of this problem will be explained in a forthcoming paper where we will give proofs of theorems in § 2.

Trivial solution. If Q is radially symmetric, i.e., $Q=Q_0(r)$ $(r=(x^2+y^2)^{1/2})$, then there exists the following trivial solution. Take a number $r_0>1$ satisfying $\pi r_0^2-\pi=\omega_0$. Then a circle γ_0 of radius r_0 with the origin as its center is a solution for any $\sigma\geqq 0$. In fact the

*) Partially supported by the Fûjukai.

corresponding stream function V is given by

$$V=V(r)=\frac{a}{\log r_0}\log r \qquad (1<r<r_0). \tag{1.5}$$

The unknown constant in (1.3) is given by

$$\frac{1}{2}\left(\frac{a}{r_0\log r_0}\right)^2+Q_0(r_0)+\frac{\sigma}{r_0}.$$

Our aim is to investigate perturbation and bifurcation of this trivial solution.

§ 2. Mathematical formulation and results. We prepare some symbols.

$$\Omega=\{(x,y)\in \boldsymbol{R}^2\,;\,1<x^2+y^2<\infty\},$$
$$S^1=\{(x,y)\in \boldsymbol{R}^2\,;\,1=x^2+y^2\},$$
$$C^{m+\alpha}(\bar{\Omega}),\ C^{m+\alpha}(S^1) \qquad (m=0,1,2,\cdots,0<\alpha<1)\,;$$

the Hölder spaces with the usual norm $\|\ \|_{m+\alpha,\Omega}$ or $\|\ \|_{m+\alpha,S^1}$, resp. We fix a number $\alpha\in\,]0,1[$ and a $Q_0\in C^{2+\alpha}([1,\infty))$. The case of $Q_0(r)=-g/r$ with a positive constant g or $Q_0\equiv 0$ is physically typical.

When a small $u\in C^{3+\alpha}(S^1)$ is given, we denote by γ_u a closed Jordan curve which is parametrized in the polar coordinates as $(r_0+u(\theta),\theta)$ $(0\leqq\theta<2\pi)$. Hereafter we identify a function on S^1 with a 2π-periodic function on $\boldsymbol{R}$. We denote a domain between Γ and γ_u by Ω_u. The curvature of γ_u is denoted by K_u. V_u is the unique solution of the Dirichlet problem

$$\Delta V_u=0 \qquad \text{in } \Omega_u, \tag{2.1}$$
$$V_u|_\Gamma=0, \qquad V_u|_{\gamma_u}=a. \tag{2.2}$$

For $u\in C^{3+\alpha}(S^1)$, $Q\in C^{2+\alpha}(\bar{\Omega})$ and $\xi\in\boldsymbol{R}$, we put

$$F_1(a,Q\,;u,\xi)=\left(\frac{1}{2}|\nabla V|^2+Q\right)\Big|_{\gamma_u}+\sigma K_u-\xi_0-\xi, \tag{2.3}$$

where

$$\xi_0=\frac{1}{2}\left(\frac{a}{r_0\log r_0}\right)^2+Q_0(r_0)+\sigma/r_0,$$

$$F_2(a,Q\,;u,\xi)=\frac{1}{2}\int_0^{2\pi}(r_0+u(\theta))^2d\theta-\pi-\omega_0, \tag{2.4}$$

$$F(a,Q\,;u,\xi)=(F_1(a,Q\,;u,\xi),\ F_2(a,Q\,;u,\xi)). \tag{2.5}$$

Then it is easy to see that $F(a,Q_0\,;0,0)=(0,0)$ and that γ_u is a solution for Q if and only if $F(a,Q\,;u,\xi)=(0,0)$ for some $\xi\in\boldsymbol{R}$. Note that $F(a,\cdot\,;\cdot,\cdot)$ is a well-defined continuous mapping from a neighborhood of $(Q_0\,;0,0)$ in $C^{2+\alpha}(\bar{\Omega})\times C^{3+\alpha}(S^1)\times\boldsymbol{R}$ into $C^{1+\alpha}(S^1)\times\boldsymbol{R}$.

Now perturbation of the trivial solution is possible in the following sense.

Theorem 1. *Assume that* $\sigma>0$. *Define* a_n *by*

$$a_n=\left(\frac{\sigma(n^2-1)/r_0^2+(\partial Q_0/\partial r)(r_0)}{r_0^{-1}+nr_0^{-1}(r_0^n+r_0^{-n})/(r_0^n-r_0^{-n})}\right)^{1/2}r_0\log r_0$$

for

$$n \in \tilde{N}=\left\{n \in N;\ \sigma(n^2-1)/r_0^2+\frac{\partial Q_0}{\partial r}(r_0)\geqq 0\right\}.$$

Then, for any $a \notin \{a_n\}_{n\in\tilde{N}}$, *there exists a positive constant* δ *such that for any* $Q \in C^{2+\alpha}(\bar{\Omega})$ *satisfying* $\|Q-Q_0\|_{2+\alpha,\Omega}<\delta$ *we have a solution* $\{u, \xi\}$ *of the equation* $F(a, Q; u, \xi)=(0, 0)$. *The solution is unique in some neighborhood of the origin.*

Theorem 2. *Assume that* $\sigma=0$. *Let* $Q_0 \in C^{19+\alpha}(\bar{\Omega})$ *satisfy*

$$\frac{\partial Q_0}{\partial r}(r_0)<a^2/r_0^3(\log r_0)^2. \tag{2.6}$$

Then there exists a positive constant ε *such that for any* $Q \in C^{10+(1/2)+\alpha}(\bar{\Omega})$ *satisfying* $\|Q-Q_0\|_{10+(1/2)+\alpha}<\varepsilon$ *we have a solution* $\{u, \xi\} \in C^{6+\alpha}(S^1)\times \boldsymbol{R}$ *of the equation* $F(a, Q; u, \xi)=(0, 0)$.

The following two theorems are concerned with bifurcation phenomena.

Theorem 3. *Fix a natural number* n. *Assume that* $\sigma>0$. *Assume also that*

$$a_m \neq a_n \qquad \text{for all } m\neq n. \tag{2.7}$$

Then there exists a branch of non-trivial solution of $F(a, -g/r; u, \xi)=(0, 0)$ *through* $(a_n, 0, 0)$. *If* n *is sufficiently large, then the bifurcation occurs subcritically.*

Theorem 4. *Assume that* $\sigma=0$. *Assume also* (2.6). *Then, in some* $C^{1+\alpha}$*-neighborhood of* γ_0, *there exists no solution other than* γ_0.

Remark. The condition (2.7) is satisfied if n is sufficiently large or if $2\sigma \geqq g$.

Theorem 1 is proved by a classical implicit function theorem, while a Nash-Moser implicit function theorem is used to prove Theorem 2. Theorem 3 is a consequence of a bifurcation theory due to Sattinger, Golubitsky and Schaeffer. Theorem 4 is proved by the maximum principle. Details will be published in the future.

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