96. *Hadamard's Variational Formula for the Bergman Kernel*

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§ 1. **Statement of the theorem.** Our purpose is to justify the Hadamard's (first) variational formula for the Bergman kernel associated to a strictly pseudo-convex domain in C^n with $n\geqq 2$.

Let $\Omega_0\subset C^n$ with $n\geqq 1$ be a bounded domain with smooth boundary $\partial\Omega_0$, given by $\Omega_0=\{z\in C^n\,;\,r(z)<0\}$, where $r\in C^\infty(C^n\,;\,R)$ with $dr\neq 0$ on $\partial\Omega_0$ is a defining function of Ω_0. Every domain close to Ω_0 is parametrized by a small real-valued function ρ on $\partial\Omega_0$ in such a way that the boundary of that domain Ω_ρ is given by

$$(1)\qquad \partial\Omega_\rho=\{z+\rho(z)\nu(z)\,;\,z\in\partial\Omega_0\},$$

where $\nu(z)=dr(z)/|dr(z)|$ is identified with an element of C^n.

Let $K^\rho(z,w)$ for $(z,w)\in\Omega_\rho\times\Omega_\rho$ denote the Bergman kernel associated to Ω_ρ, which is the reproducing kernel associated to the space $L^2H(\Omega_\rho)$ of $L^2(\Omega_\rho)$-holomorphic functions. With $\delta\rho\in C^\infty(\partial\Omega_0\,;\,R)$ and $(z,w)\in\Omega_0\times\Omega_0$ fixed arbitrarily, we set

$$(2)\qquad \delta K^\rho(z,w)=\frac{d}{d\varepsilon}K^{\rho+\varepsilon\delta\rho}(z,w)|_{\varepsilon=0},$$

which is the Hadamard's first variation of $K^\rho(z,w)$ at ρ in the direction $\delta\rho$. In the case $n=1$, it has been observed by Schiffer [10] (see also Bergman-Schiffer [2], [3]) that the variation (2) at $\rho=0$ is given by the following so-called Hadamard's (first) variational formula:

$$(3)\qquad -\delta K^0(z,w)=\int_{\partial\Omega_0}K^0(z,\zeta)K^0(\zeta,w)\delta\rho(\zeta)dS^0(\zeta),$$

where $dS^0(\zeta)$ stands for the standard surface element of $\partial\Omega_0$ at ζ. Our purpose is to prove the following:

Theorem. *If Ω_0 is strictly pseudo-convex, then the variation* (2) *at $\rho=0$ exists and is given by* (3).

Notice that the right hand side of (3) makes sense, for if Ω_0 is strictly pseudo-convex then $K^0(\cdot,\cdot)$ is smooth on $(\bar\Omega_0\times\bar\Omega_0)\backslash\varDelta$, where $\varDelta$ denotes the diagonal of $\partial\Omega_0\times\partial\Omega_0$ (see Kerzman [9]).

§ 2. **Existence of the variation (2).** We begin with constructing a diffeomorphism $e_\rho: C^n\to C^n$ for ρ small, which satisfies

$$(4)\qquad e_\rho(z)=z+\rho(z)\nu(z)\qquad \text{for } z\in\partial\Omega_0\quad (\text{cf. (1)}).$$

Given a small constant $\varepsilon_0>0$, we set $N(\varepsilon_0)=\{z\in C^n\,;\,|r(z)|<\varepsilon_0\}$. Then, every point $z\in N(\varepsilon_0)$ is uniquely expressed as $z=z_b+r(z)\nu(z_b)$, where

$z_b \in \partial\Omega_0$ is the nearest point to z. Given a constant ε_1 with $0<\varepsilon_1<\varepsilon_0/4$, we choose $\chi_0 \in C_0^\infty(\boldsymbol{R}\,;\boldsymbol{R})$ satisfying

$$\begin{matrix} \chi_0(r)=1 & \text{for } |r|\leqq \varepsilon_1, \\ \chi_0(r)=0 & \text{for } |r|\geqq 3\varepsilon_1, \end{matrix} \quad \text{and} \quad \left|\frac{d}{dr}\chi_0(r)\right| \leqq \frac{3}{4\varepsilon_1} \quad \text{for } r \in \boldsymbol{R}.$$

Given $\rho \in V(\varepsilon_1)=\{\rho \in C^\infty(\partial\Omega_0\,;\boldsymbol{R})\,;\,|\rho(z)|<\varepsilon_1 \text{ for } z \in \partial\Omega_0\}$, we define a mapping $e_\rho : \boldsymbol{C}^n \to \boldsymbol{C}^n$ by setting

$$\begin{aligned} e_\rho(z) &= z+\chi_0(r(z))\rho(z_b)\boldsymbol{\nu}(z_b) && \text{for } z \in N(\varepsilon_0), \\ e_\rho(z) &= z && \text{otherwise.} \end{aligned}$$

Then, e_ρ is a diffeomorphism satisfying (4) and $e_\rho(\Omega_0)=\Omega_\rho$.

By means of e_ρ, one can pull back in general a function f^ρ on Ω_ρ and a linear operator L^ρ acting on f^ρ as follows:

$$f_\rho = e_\rho^* f^\rho = f^\rho \circ e_\rho, \qquad L_\rho f_\rho = (e_\rho^* L^\rho e_\rho^{-1*}) f_\rho = (L^\rho(f_\rho \circ e_\rho^{-1})) \circ e_\rho.$$

Let $P^\rho : L^2(\Omega_\rho) \to L^2H(\Omega_\rho) \subset L^2(\Omega_\rho)$ denote the Bergman projection associated to Ω_ρ, which is the orthogonal projection to $L^2H(\Omega_\rho)$ and is related to $K^\rho(z, w)$ by

$$P^\rho f^\rho(z) = \int_{\Omega_\rho} K^\rho(z, w) f^\rho(w)\, dV^0(w) \qquad \text{for } f^\rho \in L^2(\Omega_\rho),$$

where $dV^0(w)$ stands for the standard volume form of $\boldsymbol{C}^n$ at $w \in \Omega_\rho$. Then, $P_\rho = e_\rho^* P^\rho e_\rho^{-1*}$ satisfies

$$P_\rho f_\rho(z) = \int_{\Omega_0} K_\rho(z, w) f_\rho(w)\, dV^0(e_\rho(w)) \qquad \text{for } f_\rho \in L^2(\Omega_0),$$

where we have set

$$(5) \qquad K_\rho(z, w) = K^\rho(e_\rho(z), e_\rho(w)) \qquad \text{for } (z, w) \in \Omega_0 \times \Omega_0.$$

Observe that if $z, w \in \Omega_0 \backslash N(\varepsilon_0)$ then $e_\rho(z)=z$, $e_\rho(w)=w$, so that $K_\rho(z, w) = K^\rho(z, w)$. Therefore, the variation (2) exists for $z, w \in \Omega_0 \backslash N(\varepsilon_0)$ provided that $K_\rho(z, w)$ depends smoothly on ρ, as far as ρ is small with respect to the $C^\infty(\partial\Omega_0)$-topology.

To see the smooth dependence of $K_\rho(z, w)$ on ρ small, we first recall the following formula due to Kerzman [9] and Bell [1]:

$$K^\rho(z, w) = P^\rho \varphi_w(z) \qquad \text{for } z, w \in \Omega_\rho, \quad \varphi_w(z) = \varphi(z-w),$$

where $\varphi \in C_0^\infty(\boldsymbol{C}^n\,;\boldsymbol{R})$ is a radially symmetric function satisfying

$$\int_{\boldsymbol{C}^n} \varphi\, dV^0 = 1, \quad \varphi(\zeta)=0 \quad \text{for } |\zeta| \geqq \varepsilon_2 \quad \text{with} \quad 0<\varepsilon_2<\operatorname{dist}(w, \partial\Omega_\rho).$$

If ε_2 is chosen so small that $0<\varepsilon_2<\varepsilon_0/4$, then

$$(6) \qquad K_\rho(z, w) = P_\rho \varphi_w(z) \qquad \text{for } z, w \in \Omega_0 \backslash N(\varepsilon_0).$$

We next recall the assumption that Ω_0 is strictly pseudo-convex. Then, if $\rho \in V(\varepsilon_1)$ is small with respect to the $C^2(\partial\Omega_0)$-topology, say

$$\rho \in V_2 = \{\rho \in V(\varepsilon_1)\,;\,|\rho|_{C^2(\partial\Omega_0)} < \varepsilon_3\} \quad \text{with} \quad \varepsilon_3>0 \text{ small},$$

then Ω_ρ is strictly pseudo-convex uniformly in $\rho \in V_2$ in the sense that

$$\sum_{i,j=1}^n \frac{\partial^2 r^\rho(z)}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j \geqq C \sum_{i=1}^n |\xi_i|^2, \qquad \text{whenever } \sum_{i=1}^n \frac{\partial r^\rho(z)}{\partial z_i} \xi_i = 0,$$

holds for each $z \in \partial\Omega_\rho$, where $r^\rho = r \circ e_\rho^{-1}$ is a defining function of Ω_ρ and $C>0$ is a constant independent of $\rho \in V_2$. In this case, the following formula due to Kerzman [9] holds:

(7) $$P^\rho = 1 - \vartheta N^\rho \bar\partial, \quad \text{thus} \quad P_\rho = 1 - \vartheta_\rho N_\rho \bar\partial_\rho,$$

where ϑ denotes the formal adjoint of $\bar\partial$, and N^ρ stands for the $\bar\partial$-Neumann operator acting on $(0,1)$-forms on Ω_ρ. The definition of ϑ_ρ, N_ρ and $\bar\partial_\rho$ will be clear. The smooth dependence of the pull back N_ρ of N^ρ on ρ small in the $C^\infty(\partial\Omega_0)$-topology has been proved by Hamilton [8] *via* the Nash-Moser process. Hence, by virtue of (5), (6) and (7), the variation (2) makes sense.

Remark 1. Another result on the stability of the Bergman kernel has been obtained recently by Greene and Krantz [5], [6], [7].

Remark 2. In the case $n=1$, the $\bar\partial$-Neumann operator reduces to the Green operator for the zero Dirichlet problem (see, e.g., [4]), and the formula (7) is still valid. Thus, the variation of the Bergman kernel is expressed by using the Green function (see [10], [2], [3]).

§ 3. Proof of the variational formula (3). Pick $(z, w) \in \Omega_0 \times \Omega_0$ arbitrarily and choose $\varepsilon_0 > 0$ so small that $z, w \in \Omega_0 \backslash N(\varepsilon_0)$. Then $K_\rho(z, w) = K^\rho(z, w)$ for $\rho \in V_2$. By the reproducing property for the Bergman kernel, we have

$$K_\rho(z, w) = K^\rho(z, w) = \int_{\Omega_0} K_\rho(z, \zeta) K_\rho(\zeta, w) J[e_\rho](\zeta) dV^0(\zeta),$$

where $J[e_\rho](\zeta)$ stands for the Jacobian determinant of the mapping e_ρ at $\zeta \in \Omega_0$. Taking the variation at $\rho=0$ in the direction $\delta\rho \in C^\infty(\partial\Omega_0; \boldsymbol{R})$, we get

(8) $$\begin{aligned} \delta K^0(z, w) = \delta K_0(z, w) &= \frac{d}{d\varepsilon} K_{\varepsilon\delta\rho}(z, w)\Big|_{\varepsilon=0} \\ &= \int_{\Omega_0} \{(I_1) + (I_2) + (I_3)\} dV^0(\zeta), \end{aligned}$$

where

$$(I_1) = \delta K_0(z, \zeta) K^0(\zeta, w), \qquad (I_2) = K^0(z, \zeta) \delta K_0(\zeta, w),$$
$$(I_3) = K^0(z, \zeta) K^0(\zeta, w) \delta J[e_0](\zeta),$$

and

$$\delta J[e_0](\zeta) = \operatorname{div} \delta X_0(\zeta), \qquad \delta X_0(\zeta) = \frac{d}{d\varepsilon} e_{\varepsilon\delta\rho}(\zeta)\Big|_{\varepsilon=0}.$$

Denoting by $\partial/\partial\nu_\zeta$ the unit exterior normal vector at $\zeta \in \partial\Omega_0$, we have $\delta X_0(\zeta) = \delta\rho(\zeta)\partial/\partial\nu_\zeta$ at $\zeta \in \partial\Omega_0$, and

$$\delta K_0(z, \zeta) = \delta K^0(z, \zeta) + \delta X_0(\zeta) K^0(z, \zeta) \qquad \text{for } \zeta \in \Omega_0,$$
$$\delta K_0(\zeta, w) = \delta K^0(\zeta, w) + \delta X_0(\zeta) K^0(\zeta, w) \qquad \text{for } \zeta \in \Omega_0,$$

where $\delta X_0(\zeta)$ is acting as a differential operator. Notice that $\delta K^0(\cdot, \cdot)$ is sesqui-holomorphic as well as $K^0(\cdot, \cdot)$, and that $K^0(\cdot, \cdot)$ is hermitian symmetric with the reproducing property. Hence,

$$\int_{\Omega_0} (I_1) dV^0(\zeta) = \delta K^0(z, w) + \int_{\Omega_0} \delta X_0(\zeta) K^0(z, \zeta) \cdot K^0(\zeta, w) dV^0(\zeta),$$
$$\int_{\Omega_0} (I_2) dV^0(\zeta) = \delta K^0(z, w) + \int_{\Omega_0} K^0(z, \zeta) \cdot \delta X_0(\zeta) K^0(\zeta, w) dV^0(\zeta),$$

while by integrating by parts,

$$\int_{\Omega_0}(I_3)dV^0(\zeta)=-\int_{\Omega_0}\delta X_0(\zeta)\{K^0(z,\zeta)K^0(\zeta,w)\}dV^0(\zeta)$$
$$+\int_{\partial\Omega_0}K^0(z,\zeta)K^0(\zeta,w)\delta\rho(\zeta)dS^0(\zeta).$$

Therefore, by (8), we obtain the desired variational formula (3).

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