88. *Tensor Products of Singular Holomorphic Representations of $SU(n, n)$ and $Mp(n, R)$*

By Hirofumi YAMADA*) and Toshiharu SUGAWARA**)

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0. Introduction. General theory of tensor products of holomorphic discrete series representations and some of their limits for groups associated with the Hermitian symmetric spaces was established by Jakobsen and Vergne in [3] and Repka in [10]. Some concrete computations of irreducible decomposition of these tensor products are carried out in [3] and [4].

We shall compute in this note the irreducible decomposition of the tensor products of representations "beyond the limits" of holomorphic discrete series of some groups. We restrict our attention only to the group $SU(n, n)$ and the two-fold covering group $Mp(n, \boldsymbol{R})$ of $Sp(n, \boldsymbol{R})$. (See below.) For these groups the irreducible representations of maximal compact subgroups are parametrized by the Young diagrams. Our computation will be reduced to that of Young diagrams. Proofs are done essentially along the line of Jakobsen's proof for a special case in [5]. Details are omitted here.

1. Holomorphic representations of $SU(n, n)$ and $Mp(n, R)$. Let

$$G_1 = SU(n, n) = \left\{ g \in SL(2n, C)\,;\; g\left(\begin{array}{c|c} & -1 \\ \hline 1 & \end{array}\right) g^* = \left(\begin{array}{c|c} & -1 \\ \hline 1 & \end{array}\right) \right\}$$

$$G_2' = Sp(n, \boldsymbol{R}) = \left\{ g \in GL(2n, \boldsymbol{R})\,;\; g\left(\begin{array}{c|c} & -1 \\ \hline 1 & \end{array}\right) {}^t g = \left(\begin{array}{c|c} & -1 \\ \hline 1 & \end{array}\right) \right\}$$

and G_2 be the metaplectic group $Mp(n, \boldsymbol{R})$, the two-fold covering group of G_2'. Let K_1 and K_2 be the maximal compact subgroups of G_1 and G_2 respectively. The elements of K_1 are complex matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $((a+\sqrt{-1}b), (a-\sqrt{-1}b)) \in S(U(n)\times U(n))$, and those of K_2 are real matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a+\sqrt{-1}b \in U(n)$. Let $u=a+\sqrt{-1}b$ and $v=a-\sqrt{-1}b$. We use the unbounded realization D_i of G_i/K_i:

$$D_1 = \{z = x+\sqrt{-1}y\,;\; x \text{ and } y \text{ are complex } n\times n \text{ matrices},$$
$$x^*=x,\ y^*=y, \text{ any } y \text{ is positive definite}\},$$

and

*) Department of Mathematics, Hiroshima University.
**) Department of Mathematics, Waseda University. Present address: N.T.T., Electrical Communication Laboratories.

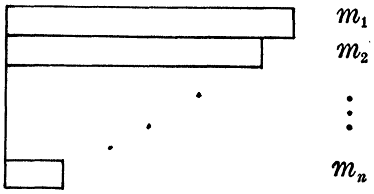
$$D_2=\{z=x+\sqrt{-1}y\,;\ x \text{ and } y \text{ are real } n\times n \text{ matrices},\ {}^t x=x,\ {}^t y=y, \text{ and } y \text{ is positive definite}\}.$$

Let τ be an irreducible unitary representation of K_i ($i=1$ or 2) on a finite dimensional vector space V_τ. Corresponding to τ there is a representation U_τ of G_i, holomorphically induced from τ, which acts on the space $\mathcal{O}(D_i, V_\tau)$ of holomorphic functions $f: D_i\to V_\tau$ by

$$(U_\tau(g)f)(z)=J_\tau(g^{-1}, z)^{-1}f(g^{-1}\cdot z).$$

If $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G_i$, $g\cdot z=(az+b)(cz+d)^{-1}$, and if $k\in K_i$, $J_\tau(k,\sqrt{-1})=\tau(k)$. If the representation U_τ is unitary on a space $H_\tau\subset\mathcal{O}(D_i, V_\tau)$, then H_τ is a reproducing kernel Hilbert space (cf. [1]).

Irreducible representations of K_i are parametrized by highest weights or Young diagrams. Let $Y=(m_1, \cdots, m_n)$ be a Young diagram



We write $|Y|=m_1+\cdots+m_n$. Let τ_Y be the corresponding polynomial representation of $U(n)$ (cf. [2]). τ'_Y denotes the representation contragredient to τ_Y.

The following proposition is proved in [6]–[8].

Proposition. *Let Y_1, Y_2 be Young diagrams and γ a positive integer. Let $U_1(Y_1, Y_2, \gamma)$ be the representation of G_1, holomorphically induced by the representation*

$$K_1\cong S(U(n)\times U(n))\ni (u, v)\longmapsto (\det u)^\gamma \tau_{Y_1}(u)\otimes\tau'_{Y_2}(v).$$

Then $U_1(Y_1, Y_2, \gamma)$ is unitary if and only if

$$Y_1=(m_1, \cdots, m_i, 0, \cdots 0), \qquad Y_2=(n_1, \cdots, n_j, 0, \cdots 0)$$

with $i+j\leqq\gamma$. Let $U_2(Y, \gamma)$ be the representation of G_2, holomorphically induced from the representation

$$K_2\cong U(n)\ni u\longmapsto (\det u)^{\gamma/2}\tau_Y(u).$$

Then $U_2(Y, \gamma)$ is unitary if and only if

$$Y=(\underbrace{m_1, \cdots, m_j, 1, \cdots, 1}_{i}, 0, \cdots, 0) \qquad \text{with } m_j\geqq 2 \text{ and } i+j\leqq\gamma.$$

For $k=1, \cdots, n-1$, let $U_{1,k}$ be the representation of G_1 holomorphically induced from $(u, v)\mapsto(\det u)^k$, and let $U_{2,k}$ be the representation of G_2 holomorphically induced from $u\mapsto(\det u)^{k/2}$. Precisely,

$$(U_{i,k}(g)f)(z)=\det(cz+d)^{-k/i}f((az+b)(cz+d)^{-1})$$

for $g^{-1}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G_i$. $U_{i,k}$ is unitary on the reproducing kernel Hilbert space $H_{i,k}$ consisting of holomorphic solutions to $F_j(\partial/\partial z)f=0$ for $j>k$ where $F_j(z)$ denotes any j-order subdeterminant of z (cf. [11]).

2. Results. We define the following set of Young diagrams:

$$I(r, k)=\{Y=(m_1, \cdots, m_k, 0, \cdots, 0)\,;\, |Y|=r\}.$$

Theorem. *For $k=1, \cdots, n-1$, we have*

$$U_{1,k}\otimes U_{1,k}=\bigoplus_{r=0}^{\infty} \bigoplus_{Y\in I(r,k)} U_1(Y, Y, 2k)$$

and

$$U_{2,k}\otimes U_{2,k}=\bigoplus_{r=0}^{\infty} \bigoplus_{Y\in I(r,k)} U_2(2Y, 2k)$$

where $2Y=(2m_1, \cdots, 2m_k, 0, \cdots, 0)$ *for* $Y=(m_1, \cdots, m_k, 0, \cdots, 0) \in I(r, k)$.

More generally, the decomposition of $U_1(Y_1, Y_2, k)\otimes U_1(Y_1', Y_2', k)$ and $U_2(Y, k)\otimes U_2(Y', k)$ $(1\leqq k\leqq n-1)$ can be calculated, but it would be a little bit complicated to express the results. As an example, we shall decompose $U_1(Y_1, \phi, k)\otimes U_1(Y_2, \phi, k)$. $(\phi=(0, \cdots, 0), 1\leqq k\leqq n-1)$. We assume that $Y_i=(m_1^i, \cdots, m_{j_i}^i, 0, \cdots, 0)$ with $j_i\leqq k$ for $i=1, 2$. For a Young diagram Y the tensor product $Y_1\otimes Y_2\otimes Y$ is decomposed by the Richardson rule (cf. [9]). Let I_Y be the set of Young diagrams appearing in the decomposition of $Y_1\otimes Y_2\otimes Y$, i.e.,

$$Y_1\otimes Y_2\otimes Y=\bigoplus_{Y_\nu\in I_Y} Y_\nu.$$

Let

$$I_Y(k)=\{Y_\nu=(m_1^\nu, \cdots, m_n^\nu)\in I_Y\,;\, m_{k+1}^\nu=0\}.$$

Then

$$U_1(Y_1, \phi, k)\otimes U_1(Y_2, \phi, k)$$
$$=\bigoplus_{Y_\nu\in I_\phi} U_1(Y_\nu, \phi, 2k)\oplus \bigoplus_{r=1}^{\infty} \bigoplus_{Y;|Y|=r} \bigoplus_{Y_\nu\in I_Y(k)} U_1(Y_\nu, Y, 2k).$$

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