78. On Integral Transformations Associated with a Certain Riemannian Metric

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§1. Statement of the result. Let (M, g) be a complete, connected and simply connected Riemannian manifold of $\dim M=m$. We consider the following integral transformation with a parameter $t>0$.

$$(H_t f)(x)=(2\pi t)^{-m/2}\int_M \rho(x,y)e^{-d^2(x,y)/2t}f(y)d_g(y),$$

where $d_g(y)=g(y)^{1/2}dy$, $g(y)=\det g_{ij}(y)$, $d(x,y)$ denotes the Riemannian distance between x, y and $\rho(x,y)=|\det(d\,\mathrm{Exp}_x^{-1})_y|^{1/2}$ with Exp_x standing for the exponential mapping at x.

We assume the following:

(A.1) (M, g) has a non-positively pinched sectional curvature, i.e. there exists a constant $k>0$ such that for any 2-plane π, the sectional curvature K_π satisfies $-k^2\leqq K_\pi\leqq 0$.

(A.2) There exist constants C_1, C_2 such that for any x, y and $z\in M$, we have

$$|\Delta^{(z)}\rho(x,z)|\leqq C_1,$$
$$|\Delta^{(z)}\rho(x,z)-\Delta^{(z)}\rho(y,z)|\leqq C_2 d(x,y)$$

where $\Delta^{(z)}$ is the Laplace-Beltrami operator acting on a function of z, i.e.,

$$\Delta^{(z)}f(z)=g(z)^{-1/2}\sum_{i,j=1}^{m}(\partial/\partial z^i)(g(z)^{1/2}g^{ij}(z)(\partial f(z)/\partial z^j)).$$

Theorem. *Let (M, g) be a Riemannian manifold satisfying above conditions. Then, we have the following for an arbitrary number $T>0$.*

(a) *The integral transformation H_t defines a bounded linear operator in $L^2(M, d_g)$ for $0<t<T$.*

(b) $s-\lim_{t\to 0+} H_t f=f$ *for* $f\in L^2(M, d_g)$.

(c) *There exists a constant C_3 such that*

$$\|H_{t+s}f-H_tH_sf\|\leqq C_3((t+s)^{3/2}-t^{3/2}+s^{3/2})\|f\|$$

for $0<t, s, t+s<T$ and $f\in L^2(M, g)$.

(d) *There exists a limit in operator norm $\lim_{k\to\infty}(H_{t/k})^k$ for any $t>0$, denoted by $\boldsymbol{H}_t$, which forms with $\boldsymbol{H}_0=Id$ a C^0-semi group in*

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$L^2(M, d_g)$ whose infinitesimal generator is given by

$$\partial(H_t f)(x)/\partial t|_{t=0} = ((1/2)\Delta - (1/12)R(x))f(x) \qquad \text{for } f \in C_0^\infty(M)$$

where $R(x)$ is the scalar curvature associated with the Riemannian metric g.

Remarks. (1) In our case, $d^2(x, y)/2t$ equals to $S(t; x, y)$ where

$$\begin{cases} S(t; x, y) = \inf_\gamma \int_0^t L(\gamma(\tau), \dot\gamma(\tau))d\tau \,;\, \gamma \in C_{t;x,y}, \qquad (\dot\gamma(\tau) = (d\gamma(\tau)/d\tau)), \\ C_{t;x,y} = \{\gamma(\cdot) \in C([0, t] \to M) : \text{absolutely continuous in } \tau \\ \qquad\qquad\qquad\qquad \text{with } \gamma(0) = y,\ \gamma(t) = x\}, \\ L(\gamma, \dot\gamma) = 2^{-1} \sum_{i,j=1}^m g_{ij}(\gamma)\dot\gamma^i\dot\gamma^j \qquad \text{for } (\gamma, \dot\gamma) \in T_x M. \end{cases}$$

(2) If (M, g) is a manifold of the constant negative curvature, then the function ρ satisfies conditions of (A.2).

(3) If we replace ρ with 1, then it seems difficult to prove (c) and (d). We are not sure whether it is possible or not.

Detailed proof will appear in the forthcoming paper.

§ 2. Sketch of the proof.

Put $k(t; x, y) = (2\pi t)^{-m/2}\rho(x, y)e^{-d^2(x,y)/2t}$.

Lemma 1. *Under Assumption* (A.1), *there exists a constant C_4 independent of $x, y \in M$ such that for any $0 < t < T$, we have*

$$\int_M k(t; x, y)d_g(x) \leqq C_4 \quad \text{and} \quad \int_M k(t; x, y)d_g(y) \leqq C_4.$$

To prove these, we use the normal polar coordinate at y or x. By using Rauch's comparison theorem, we may estimate the kernel $k(t; x, y)$ appropriately. Combining this with Schwarz's inequality, we prove (a) easily (see Kobayashi-Nomizu [3] for geometrical terms).

To prove (b) under Assumption (A.1), we take $f \in C_0^\infty(M)$. Introducing the cut off function $\chi(x)$, $\chi(x) = 1$ for $d(x, \text{supp } f) \leqq 2L$ and $= 0$ for $d(x, \text{supp } f) > 2L$ with L fixed suitably, we may prove the following:

Lemma 2. *For $f \in C_0^\infty(M)$, we have*

$$(*) \qquad \lim_{t\to 0+} \|\chi(H_t f) - f\| = 0$$

$$(**) \qquad \lim_{t\to 0+} \|(1-\chi)H_t f\| = 0.$$

$(*)$ follows from direct computations for any L. By using the method of oscillatory integrals for sufficiently large L, we prove $(**)$. Analogous procedure was already presented in Fujiwara [2].

For any $0 < t, s, t+s < T$ and $f \in C_0^\infty(M)$, we have

$$(H_t H_s f)(x) - (H_{t+s} f)(x) = \int_M h(t, s; x, y)f(y)d_g(y),$$

where

$$h(t, s; x, y) = \int_0^s \left\{\frac{d}{d\sigma}\int k(t+s-\sigma; x, z)k(\sigma; z, y)d_g(z)\right\}d\sigma.$$

Lemma 3. *Under Assumptions* (A.1) *and* (A.2), *we have*

$$h(t, s; x, y)=\int_0^s d\sigma\Big[(2\pi(t+s-\sigma))^{-m/2}(2\pi\sigma)^{-m/2}\int_M (\Delta^{(z)}\rho(x, z)-\Delta^{(z)}\rho(y, z)) \times e^{-(d^2(x,z)/2(t+s-\sigma))-(d^2(y,z)/2\sigma)}d_g(z)\Big].$$

Moreover, there exists a constant C_5 *such that for any* $0<t, s, t+s<T$, *we have*

$$\int_M |h(t, s; x, y)|\, d_g(y)\leqq C_5((t+s)^{3/2}-t^{3/2}+s^{3/2})$$

$$\int_M |h(t, s; x, y)|\, d_g(x)\leqq C_5((t+s)^{3/2}-t^{3/2}+s^{3/2}).$$

Rewrite the above integrals in normal coordinate. Using Toponogov's comparison theorem and Hamilton-Jacobi equation for $S(t; x, y)$, we have the desired results.

From above lemma, we may prove (c) and construct $H_t f$ by using the analogous method in Fujiwara [2]. See also Chernoff [1].

Lemma 4. *Under Assumptions* (A.1) *and* (A.2), *we have, for any* $f\in C_0^\infty(M)$,

$$\frac{\partial}{\partial t}(H_t f)(x)=(1/2)(H_t\Delta f)(x)-(1/2)(H_t Rf)(x)+(G_t f)(x),$$

where

$$(G_t f)(x)=(2\pi t)^{-m/2}\int_M ((1/2)\Delta^{(x)}\rho(x, y)-(R(x)/12)) \times e^{-d^2(x,y)/2t}f(y)d_g(y).$$

Moreover, we have the estimate $\|G_t f\|=O(t^{1/2})\|f\|$ *for* $f\in C_0^\infty(M)$.

There are shown by direct computations combined with the fact $\Delta^{(x)}\rho(x, y)|_{y=x}=R(x)/6$. This lemma combined with (b) gives (d).

References

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[3] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry. Wiley, New York (1963).