41. *Uniqueness and Non Uniqueness in the Cauchy Problem for a Class of Operators of Degenerate Type*

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In this paper, we extend Calderón's uniqueness theorem in the non characteristic Cauchy problem (see Nirenberg [2]) to a certain class of operators whose characteristic roots degenerate on the initial surfaces. We also extend Plis' result on non uniqueness to a degenerate elliptic operator. Detailed proofs will be published elsewhere.

Statement of results. Let U be a neighborhood of 0 in $\boldsymbol{R}^{n+1}=\boldsymbol{R}_t\times\boldsymbol{R}^n_x$. And let $P=P(t,x;D_t,D_x)$ be a partial differential operator of order m with C^∞-coefficients in U. Here $D_t=\partial/i\partial t$, $D_x=\partial/i\partial x$. We assume the following conditions.

(A.1) The principal symbol $P_m(t,x;\tau,\xi)$ of P is factorized as

$$P_m(t,x;\tau,\xi)=\prod_{j=1}^{s}(\tau-t^l\lambda_j(t,x;\xi))^2\prod_{k=s+1}^{m-s}(\tau-t^l\lambda_k(t,x;\xi)),$$

where l is a positive integer and $\lambda_j(t,x;\xi)$ $(1\leqq j\leqq m-s)$ are C^∞-functions in $U\times(\boldsymbol{R}^n\backslash 0)$, homogeneous of degree 1 in ξ. We require that λ_j satisfy Calderón's conditions in $U\times(\boldsymbol{R}^n\backslash 0)$:

(A.2) $\lambda_i\neq\lambda_j$ $(i\neq j)$,

(A.3) $\operatorname{Im}\lambda_j\neq 0$ $(1\leqq j\leqq s)$,

(A.4) $\operatorname{Im}\lambda_k\neq 0$ or $\equiv 0$ $(s+1\leqq k\leqq m-s)$.

All the conditions above are imposed on the principal part of P. Now, we consider the lower order terms of P. From (A.1), we can easily see that there exist differential polynomials Q and R, homogeneous of degree s and $m-2s$ respectively such that

$$P_m(t,x;\tau,\xi)=R(t,x;\tau,t^l\xi)\cdot Q(t,x;\tau,t^l\xi)^2,$$

and Q and R have distinct characteristic roots (cf. Smith [5]). Hence we can express P as

$$P(t,x;D_t,D_x)=R(t,x;D_t,t^lD_x)\cdot Q(t,x;D_t,t^lD_x)^2+\sum_{j=1}^{m}P'_{m-j}(t,x;D_t,D_x),$$

where $P'_{m-j}(t,x;D_t,D_x)=\sum_{i=0}^{m-j}\sum_{|\alpha|=i}a_{i,j,\alpha}(t,x)D_x^\alpha D_t^{m-j-i}$, and $a_{i,j,\alpha}\in C^\infty(U)$.

(A.5) There exist $b_{i,j,\alpha}\in C^\infty(U)$ such that

$a_{i,j,\alpha}(t,x)=t^{[il-j]_+}b_{i,j,\alpha}(t,x)$, where $[k]_+=\max(k,0)$.

Note that, from the assumptions above, there exists a differential polynomial $\tilde{P}$ of degree m such that

$$t^m P(t, x; D_t, D_x) = \tilde{P}(t, x; tD_t, t^{l+1}D_x).$$

(A.6) $\sum_{i=1}^{m-1} \sum_{|\alpha|=i} b_{i,1,\alpha}(t, x)\xi^\alpha \lambda_j(t, x; \xi)^{m-1-i}|_{t=0} = 0$ $(1 \leqq j \leqq s)$.

Note that $\lambda_j(t, x; \xi)$ are characteristic roots of $\tilde{P}(t, x; D_t, D_x)$. And if we denote the subprincipal symbol of $\tilde{P}$ by $\tilde{P}^s_{m-1}$, (A.6) implies $\tilde{P}^s_{m-1}(t, x; \lambda_j(t, x; \xi), \xi)|_{t=0} = 0$ for double roots $\lambda_j(t, x; \xi)$ $(1 \leqq j \leqq s)$ of $\tilde{P}$.

Now, we state the main theorem.

Theorem 1. *Under assumptions* (A.1)–(A.6), *there exists a neighborhood U' of 0 in $\boldsymbol{R}^{n+1}$, such that if $u \in C^\infty(U)$ satisfies $Pu = 0$ in U and $(D_t^j u)(0, x) = 0$ $(0 \leqq j \leqq m-1)$, then $u = 0$ in U'.*

Remark 1. This theorem is an extension of the results of Roberts [4] and Uryu [7]. Roberts treated the case $l \leqq 0$ (i.e. Fuchsian type equations), and Uryu treated the case $s = 0$. See Tahara [6] for condition (A.5) and see [4] for condition (A.6).

Example. Let P be the operator:

$$P = (D_t - it^l D_x)^2 + a(t, x)D_t + b(t, x)D_x + c(t, x),$$

where a, b and $c \in C^\infty(U)$, U is a neighborhood of 0 in $\boldsymbol{R}^2$. Then P satisfies our conditions if $b(t, x) = t^l \tilde{b}(t, x)$ for some $\tilde{b} \in C^\infty(U)$.

As for the necessary condition for uniqueness, we consider the following example of a degenerate elliptic operator:

$$P = (\partial_t - it^l \partial_x)^p + t^k (i\partial_x)^q - t^m (i\partial_x)^{q-r},$$

where p, q, r, k and $l \in \boldsymbol{N}$, $r \leqq q \leqq p$ and $m \in \boldsymbol{Z}$, $0 \leqq m < k$.

Theorem 2. *Under the following condition* (1) *or* (2), *there exist C^∞-functions u and f in $\boldsymbol{R}^2$ such that*

$$Pu - fu = 0, \qquad 0 \in \operatorname{supp} u \subset \{t \geqq 0\}.$$

(1) *When* $p > q$,

$(1)_1$ $k - r(pl-k)/(p-q) \leqq m < k - r(k+p)/q$,

or

$(1)_2$ $q \geqq (p+1)/2, \quad k < q(l+1) - \mathrm{p}, \quad m < k - r(pl-k)/(p-q)$,

or

$(1)_3$ $\begin{cases} q > (p+1)/2, \quad k \geqq q(l+1) - p, \\ m < k + r(pl+l+1-p-2k)/(2q-p-1), \end{cases}$

or

$(1)_4$ $\begin{cases} q < (p+1)/2, \\ k + r(pl+l+1-p-2k)/(2q-p-1) < m < k - r(pl-k)/(p-q). \end{cases}$

(2) *When* $p = q$,

$(2)_1$ $k \leqq pl, \quad m < k - r(k+p)/p$,

or

$(2)_2$ $k > pl, \quad m < k + r(pl+l+1-p-2k)/(p-1)$.

Remark 2. This theorem is a slight modification of Plis [3, Theorem 4]. He treated the case $l = m = 0$, $r = 1$.

Remark 3. Condition $(2)_1$ with $k = pl$ implies $m < l(p-r) - r$. On the other hand, Theorem 1 with $s=0$ shows that uniqueness holds in this case if $m \geqq l(p-r)-r$. Hence this necessary condition seems to be the best one and, in Theorem 1, assumption (A.5) is indispensable.

References

[1] S. Alinhac and M. S. Baouendi: Uniqueness for the characteristic Cauchy problem and strong unique continuation for higher order partial differential inequalities. Amer. J. Math., **102**, 179–217 (1979).

[2] L. Nirenberg: Lectures on linear partial differential equations. Regional Conference Series in Math. (1973).

[3] A. Plis: A smooth linear elliptic differential equation without any solution in a sphere. Comm. Pure Appl. Math., **14**, 599–617 (1961).

[4] G. Roberts: Uniqueness in the Cauchy problem for characteristic operators of Fuchsian type. J. Differential Equations, **38**, 374–392 (1980).

[5] T. Smith: Some remarks on a paper of Calderón on existence and uniqueness theorems for systems of partial differential equations. Comm. Pure Appl. Math., **18**, 415–441 (1965).

[6] H. Tahara: Cauchy problems for Fuchsian hyperbolic partial differential equations. Proc. Japan Acad., **54A**, 92–96 (1978).

[7] H. Uryu: Uniqueness for the characteristic Cauchy problem and its applications (to appear).