35. A Versal Family of Hironaka's Additive Group Schemes

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In connection with the resolution of singularities of algebraic varieties in positive characteristics, Hironaka [1] considered certain subgroup schemes, now called Hironaka's additive group schemes, of the vector group over a field k of characteristic $p>0$. Oda [3] then reduced their study to linear algebra as follows, and together with Mizutani [2] classified them in low dimensions: Hironaka's additive subgroup schemes of exponent not greater than e in an n-dimensional vector group over k are in one-to-one correspondence with the pairs (V, W) consisting of an n-dimensional k^q-vector space W (with $q=p^e$) and a proper k-subspace V of $k\otimes_{k^q}W$ satisfying the condition

$$(*) \qquad \mathcal{N}_e\mathcal{D}_e(V)=V$$

(cf. [3, Theorem 2.6]). Moreover, the exponent is exactly e if and only if either $e=0$, or $e>0$ and V is not generated over k by $V\cap(k^p\otimes_{k^q}W)$.

The above condition (*), however, is rather difficult to deal with. We give below alternative formulations of (*), which enable us easily to produce examples and which, hopefully, might turn out to be theoretically useful. Some of our formulations have close connection with the one given by Russel [4].

The details will appear elsewhere.

Results. To deal with all exponents simultaneously and to describe Hironaka's additive group schemes more directly than in [3, § 2], we take the "p^{-e}-th power" of the situation above.

For a nonnegative integer e, let $k_e=k^{1/p^e}$, inside a fixed algebraic closure $\bar{k}$ of k, consist of the p^e-th roots of elements of k and let $k_\infty=\bigcup_{e\geqslant 0}k_e$ be the perfect closure of k. Let $I_\infty\subset k_\infty\otimes_k k_\infty$ (resp. $I_e\subset k_e\otimes_k k_e$) be the kernel of the multiplication map $k_\infty\otimes_k k_\infty\to k_\infty$ (resp. $k_e\otimes_k k_e\to k_e$) so that $I_\infty=\bigcup_{e\geqslant 0}I_e$. Define also the ideal $J_\infty\subset I_\infty$ to be the union

$$J_\infty=\bigcup_{e\geqslant 0}(I_e)^{pe}$$

of the p^e-th power ideals of the ideals I_e. Consider the left k_e-vector space Diff (k_e/k) of differential operators over k of k_e into itself. Then it is naturally a subset of Diff (k_{e+1}/k) via the injection sending $D\in$ Diff (k_e/k) to the operator $k_{e+1}\ni u\mapsto(D(u^p))^{1/p}\in k_{e+1}$. We can thus consider the union

$$\mathrm{Diff}\,(k_\infty/k)=\bigcup_{e\geqslant 0}\mathrm{Diff}\,(k_e/k)$$

with the nonnegative rational valued *order function*

$$\mathrm{ord}: \mathrm{Diff}\,(k_\infty/k) \longrightarrow \boldsymbol{Q}_{\geqslant 0}$$

defined as follows: if $D \in \mathrm{Diff}\,(k_e/k)$, then $\mathrm{ord}\,(D)$ is $1/p^e$ times the usual order of the differential operator D. Note that this definition is compatible with the embedding above.

For a k-vector space T, let $\mathrm{Diff}\,(k_\infty/k)$ act on $k_\infty \otimes_k T$ through the first factor. Then for a k_∞-subspace V of $k_\infty \otimes_k T$, we define k_∞-subspaces $\mathcal{D}(V)$ and $\mathcal{N}(V)$ of $k_\infty \otimes_k T$ by

$\mathcal{D}(V):=\{Dv\,;\, v \in V \text{ and } D \in \mathrm{Diff}\,(k_\infty/k) \text{ with } \mathrm{ord}\,(D)<1\}$,

$\mathcal{N}(V):=\{x \in k_\infty \otimes_k T\,;\, Dx \in V \text{ for all } D \in \mathrm{Diff}\,(k_\infty/k) \text{ with } \mathrm{ord}\,(D)<1\}$.

Then we have the following:

Theorem. *Hironaka's additive group schemes are in one-to-one correspondence with the pairs (V, T) consisting of a finite dimensional k-vector space T and a proper k_∞-subspace of $k_\infty \otimes_k T$ satisfying the following equivalent conditions* (1) *through* (4). *Moreover, a Hironaka's additive group scheme is of exponent $\leqslant e$ if and only if the corresponding V is defined over k_e,* i.e., *is generated over k_∞ by $V \cap (k_e \otimes_k T)$.*

(1) $\mathcal{N}\mathcal{D}(V)=V$.

(2) *There exists a unique k_∞-subspace U of $k_\infty \otimes_k T$ satisfying $\mathcal{D}\mathcal{N}(U)=U$ such that $V=\mathcal{N}(U)$.*

(2′) *There exists a k_∞-subspace U of $k_\infty \otimes_k T$ such that $V=\mathcal{N}(U)$.*

(3) *For the dual k-vector space T^* of T, there exists a k_∞-subspace V' of $k_\infty \otimes_k T^*$ such that $V=(\mathcal{D}(V'))^\perp$, the perpendicular with respect to the canonical pairing $(k_\infty \otimes_k T) \times (k_\infty \otimes_k T^*) \to k_\infty$ induced by the dual pairing for T and T^*.*

(3′) *For the dual k-vector space T^* of T, there exists a k_∞-subspace V' of $k_\infty \otimes_k T^*$ such that with respect to the canonical pairing $\langle\ ,\ \rangle: (k_\infty \otimes_k T) \times (k_\infty \otimes_k T^*) \to k_\infty \otimes_k k_\infty$ induced by the original dual pairing for T and T^*, we have*

$$V=\{v \in k_\infty \otimes_k T\,;\, \langle v, v' \rangle \in J_\infty \text{ for all } v' \in V'\}.$$

(4) *There exists a k_∞-subspace U of $k_\infty \otimes_k T$ such that, via the map $i: k_\infty \otimes_k T \to k_\infty \otimes_k k_\infty \otimes_k T$ defined by $i(a \otimes t)=a \otimes 1 \otimes t$ for $a \in k_\infty$ and $t \in T$, we have*

$$V=\{v \in k_\infty \otimes_k T\,;\, i(v) \in k_\infty \otimes_k U + J_\infty \otimes_k T\}.$$

Corollary 1. *Let (V, T) be a pair satisfying the equivalent conditions of Theorem. Let $R(V, T)$ be the graded k-subalgebra generated by the canonical image of T in the symmetric product* $\mathrm{Symm}_{k_\infty}((k_\infty \otimes_k T)/V)$ *over k_∞ of the k_∞-vector space $(k_\infty \otimes_k T)/V$. Then* $\mathrm{Spec}\,(R(V, T))$ *is the corresponding Hironaka's additive subgroup scheme of the vector group* $\mathrm{Spec}\,(\mathrm{Symm}_k\,(T))$.

Corollary 2. *Let $\tilde{T}_0$ be a countably infinite dimensional k-vector* *space. Regard* $\tilde{T}_\infty := k_\infty \otimes_k \tilde{T}_0$ *as a k-vector space and consider the* k_∞*-subspace* $\tilde{V} := J_\infty \otimes_k \tilde{T}_0$ *of* $k_\infty \otimes_k \tilde{T}_\infty$ *which is thought of as a* k_∞*-vector space through the first factor. Then* $(\tilde{V}, \tilde{T}_\infty)$ *is a "versal" pair for those satisfying the equivalent conditions of Theorem: a pair* (V, T) *consisting of a finite dimensional k-vector space T and a proper* k_∞*-subspace V of* $k_\infty \otimes_k T$ *satisfies the equivalent conditions of Theorem if and only if there exists a k-linear map* $\psi : T \to \tilde{T}_\infty$ *such that V is the inverse image of* $\tilde{V}$ *by* $1 \otimes \psi : k_\infty \otimes_k T \to k_\infty \otimes_k \tilde{T}_\infty$.

Remark. If we disregard the ambient vector group, then we may restrict ourselves to injective ψ's, i.e., to finite dimensional k-subspaces T of $\tilde{T}_\infty$ and $V = \tilde{V} \cap (k_\infty \otimes_k T)$. In this way, the examples in [3], [2], [4] can be described in a transparent manner. We get a similar versal pair of exponent $\leqslant e$, if we replace k_∞ and J_∞ above by k_e and $(I_e)^{p^e}$.

References

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