30. Singular Integrals on a Locally Compact Abelian Group with an Action of a Compact Group

By Kōichi SAKA
Institute of Mathematics, Akita University

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Let G be a semidirect product group of a closed normal abelian subgroup A and a compact subgroup K. An action of K on A is given by $k(a)=kak^{-1}$ for all $k\in K$ and $a\in A$. Let $\hat{A}$ be the dual group of A. If $k\in K$ and $\gamma\in\hat{A}$, the equation that $\langle k(\gamma),a\rangle=\langle\gamma,k^{-1}(a)\rangle$ for all $a\in A$, defines an action of K on $\hat{A}$. If E is a finite dimensional vector space, then for $1\leqq p\leqq\infty$, $L^p(A;E)$ will denote the Banach space of all L^p functions on A with values in E. Suppose (λ,E) is a finite dimensional unitary representation of K. A G-action $\tau(g)$ on $L^p(A;E)$ will be defined by $\tau(g)f(a')=\tau(ak)f(a')=\lambda(k)f(k^{-1}(a^{-1}a'))$ for all $a'\in A$ and all $g\in G$ with $g=ak$, $a\in A$, $k\in K$. The *Fourier transform* of f in $L^1(A;E)$ is defined by $\mathfrak{F}f(\gamma)=\int_A\langle\gamma,\overline{a}\rangle f(a)da$ for all $\gamma\in\hat{A}$. We give a definition of a polar decomposition (Σ,C) of A (cf. [3]). Let K_0 be a closed subgroup of K, and let C be a Borel subset of A whose elements are invariant under the action of K_0. Let Σ be the homogeneous space K/K_0. We say that (Σ,C) is a *polar decomposition* of A provided that

(a) for each r in C the stability group of r in K is precisely K_0, and

(b) the mapping $(kK_0,r)\mapsto k(r)$ is a homeomorphism of $\Sigma\times C$ onto a Borel subset A_0 of A whose complement in A is of Haar measure zero in A. To avoid a trivial case we assume that the identity element e of A does not belong to A_0 throughout this paper.

Let $\hat{K}$ be the set of all equivalence classes of irreducible unitary representations of K. For π in $\hat{K}$ we denote by $d(\pi)$ the dimension of π and by $m(\pi)$ the multiplicity with which π occurs in $L^2(\Sigma)$. Le $\tilde{K}$ be the subset of $\hat{K}$ consisting of all elements with $m(\pi)\neq 0$. We set $\tilde{K}_0=\tilde{K}-\{$the trivial representation$\}$. Let (π,H_π) be an element of $\tilde{K}$ and let $\{v_j^\pi\}_{j=1,\dots,d(\pi)}$ be a fixed orthonormal basis of H_π such that $\pi(k_0)v_j^\pi=v_j^\pi$, $j=1,\cdots,m(\pi)$ for all k_0 in K_0. We put $Y_{ij}^\pi(k)=\sqrt{d(\pi)}(v_i^\pi,\pi(k)v_j^\pi)$, $i=1,\cdots,d(\pi)$, $j=1,\cdots,m(\pi)$. Then the set of functions $\{Y_{ij}^\pi:\pi\in\tilde{K}$, $i=1,\cdots,d(\pi)$, $j=1,\cdots,m(\pi)\}$ is an orthonormal basis of $L^2(\Sigma)$. We call the functions Y_{ji}^π *generalized spherical harmonics* in $L^2(\Sigma)$. Throughout this paper we will use a fixed set of generalized spherical harmonics Y_{ij}^π and we assume that A and $\hat{A}$ have polar decompositions

(Σ, C) and $(\Sigma, \tilde{C})$ respectively. The following Theorems 1 and 2 can be proved by using a generalization of Theorem (3.10) in Coifman-Weiss [1, p. 40].

Theorem 1 (A generalized Bochner's theorem). *For π in $\tilde{K}$, we consider a function f in $L^2(A)$ of the form $f(k(r))=f_1(r)Y^{\pi}_{ns}(k)$, $k\in K$, $r\in C$ where f_1 is in $L^1(C)$. Then the Fourier transform of f is of the form*

$$\mathfrak{F}f(k(\xi))=\sum_{i=1}^{m(\pi)} f_i^*(\xi)Y^{\pi}_{ni}(k), \quad k\in K, \quad \xi\in\tilde{C}$$

where f_1^* is in $L^2(\tilde{C})$.

This theorem contains those of Gelbart [2] and Herz [4] in a special case.

Theorem 2 (A generalized Riesz transform I). *Let (λ, E) be in $\tilde{K}$ and $\{u_j\}$ an orthonormal basis in E. Let (λ^*, E^*) be its contragredient representation and $\{u_j^*\}$ the dual basis of $\{u_j\}$. We consider a bounded linear operator T of $L^2(A)$ to $L^2(A;E^*)$. Then the operator T is invariant under the G-action if and only if there exists a set of functions $\{c_i\}_{i=1,\cdots,m(\lambda)}$ in $L^\infty(\tilde{C})$ such that $\mathfrak{F}(Tf)(\gamma)=M(\gamma)\mathfrak{F}f(\gamma)$ for all $\gamma\in\hat{A}$ where $M(\gamma)=\sum_{j=1}^{d(\lambda)} M_j(\gamma)u_j^*$ and $M_m(k(\xi))=\sum_{i=1}^{m(\lambda)} c_i(\xi)Y^{\lambda}_{ji}(k)$ for all $k\in K$ and $\xi\in\tilde{C}$. Moreover, if the above holds, then we have*

$$\|c_i\|_\infty \leqq \sqrt{d(\lambda)} \min_{1\leqq j\leqq d(\lambda)} \|M_j\|_\infty.$$

From now we consider the case that A is an n-dimensional real vector space with an inner product and K is a compact Lie group which acts orthogonally on A and a polar decomposition (Σ, C) of A has the following additional conditions:

(c) C is a submanifold of A,

(d) the mapping $(kK_0, r)\mapsto k(r)$ of $\Sigma\times C$ onto A_0 is a diffeomorphism,

(e) A_0 is open and $a\in A_0$ implies $-a\in A_0$,

(f) if r is in C then tr is in C but $-tr$ does not belong to C for all $t>0$.

We call a diffeomorphism $r\mapsto\delta(r)$ of C into $GL(A)$ (the set of all invertible linear transformations of A) *a system of dilations* of A provided that

(a) $\delta(r)$ is symmetric with respect to the inner product of A for all r in C,

(b) for each r in C there exists an element r^{-1} in C such that $\delta(r^{-1})$ is the inverse transformation of $\delta(r)$,

(c) there exists an element 1 in C such that $\delta(1)$ is the identity transformation of A,

(d) $\det\delta(r)$ is positive for all r in C and there exists a positive number ρ such that $\det\tau(tr)=t^\rho\det\tau(r)$ for all $t>0$ and all r in C,

(e) the subset $\{r\in C: \det\tau(r)=1\}$ is compact in C.

We define a pseudo-norm $|x|$ induced by the system of dilation $\delta(r)$ in the following way:

(a) when x is in C, we set $|x|=|\det \tau(x)|^{1/\rho}$,

(b) when x is in A_0 such that $x=k(r)$, $k\in K$, $r\in C$, we set $|x|=|r|$.

(c) when x does not belong to A_0, we set $|x|=0$.

Theorem 3. *Let (λ, E) be in $\tilde{K}_0$ and let (λ^*, E^*), $\{u_j\}$ and $\{u_j^*\}$ be as in Theorem 2. Let $\Omega(x)$ be a function on A such that $\Omega(k(r)) =\sum_{j=1}^{d(\lambda)} Y_{ji}^{\lambda}(k)u_j^*$ for all k in K and r in C. We put*

$$T_\varepsilon f(x)=\int_{|y|\geqq\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy, \qquad \varepsilon>0.$$

Then, $Tf=\lim_{\varepsilon\to 0} T_\varepsilon f$ exists in L^p-norm $(1<p<\infty)$ and the operator T is a bounded linear operator of $L^p(A)$ to $L^p(A\,;E^*)$ which is invariant under the A-action. Moreover there exists a set of functions $\{c_{li}^{\lambda}: l=1,\cdots,m(\lambda), i=1,\cdots,m(\lambda)\}$ in $L^\infty(C)$ such that $\mathfrak{F}(Tf)(x)=M(x)\mathfrak{F}f(x)$ where $M=\sum_{j=1}^{d(\lambda)} M_{ji}u_j^*$ and $M_{ji}(k(r))=\sum_{l=1}^{m(\lambda)} c_{li}^{\lambda}(r)Y_{jl}^{\lambda}(k)$ for all $k\in K$ and $r\in C$ and there exists a constant B such that $\sup_{l,i}\|c_{li}^{\lambda}\|_\infty\leqq B\sqrt{d(\lambda)}$.

Theorem 4 (A generalized Riesz transform II)**.** *Let (λ, E), (λ^*, E^*), $\{u_j\}$ and $\{u_j^*\}$ be as in Theorem 3. Suppose that the system of dilations $\delta(r)$ satisfies that $\delta(r)k(1)=k(r)$ for all $r\in C$ and $k\in K$. Then a bounded linear operator $T: L^2(A)\to L^2(A\,;E^*)$ is G-action invariant and dilation invariant if and only if there exists a set of constants $\{c_i: i=1,\cdots, m(\lambda)\}$ such that $\mathfrak{F}(Tf)(x)=M(x)\mathfrak{F}f(x)$ and $M(k(r))=\sum_{j=1}^{d(\lambda)}\sum_{i=1}^{m(\lambda)} c_iY_{ji}^{\lambda}(k)u_j^*$ for all $k\in K$ and $r\in C$.*

Theorem 5. *For $1\leqq p\leqq\infty$ and a nonnegative integer l, we denote by $W_p^l(A)$ the Sobolev space with the norm $\|f\|_{p,l}$. Let $Y(x,k)$ be a function on $A\times\Sigma$ with $\int_\Sigma Y(x,k)dk=0$ such that $Y(x,k)\in C^\infty(\Sigma)$ for each $x\in A$, and $D_k^\alpha Y(x,k)\in W_\infty^l(A)$ for all $k\in K$ and all K-invariant differential operator D_k^α on Σ of order α with $|\alpha|\leqq h$ where $h=2([m/2]+1)$ $(m=\dim K)$ and l is a fixed nonnegative integer. Let $\Omega(x,y)$ be a function on $A\times A$ such that $\Omega(x,k(r))=Y(x,k)$ for all $x\in A$, all $k\in K$ and $r\in C$ and φ a function in $W_\infty^l(A)$. We define an operator T_ε by*

$$T_\varepsilon f(x)=\varphi(x)f(x)+\int_{|y|\geqq\varepsilon}\frac{\Omega(x,y)}{|y|^n}f(x-y)dy, \qquad \varepsilon>0.$$

We put $\|Y\|_l=\max_{0\leqq|\alpha|\leqq h}\{\sup_{k\in\Sigma}\|D_k^\alpha Y(x,k)\|_{\infty,l}\}$ and $B=\max\{\|\varphi\|_{\infty,l}, \|Y\|_l\}$. Then for $1<p<\infty$, we have $\|T_\varepsilon f\|_{p,i}\leqq CB\,\|f\|_{p,i}$ with a constant C for all integers $0\leqq i\leqq l$, and $\lim_{\varepsilon\to 0} T_\varepsilon f=Tf$ exists in W_p^i-norm. The operator T also satisfies $\|Tf\|_{p,i}\leqq CB\,\|f\|_{p,i}$ for all integers $0\leqq i\leqq l$ with the same constant C.

Our case in this paper contains the Euclidean space case and the matrix space case ([2]–[4]) as examples. Details of result in this paper will appear elsewhere. I would like to thank Prof. S. Igari, M. Kaneko, A. Kodama and the referee for many valuable suggestions.

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