114. *Microlocal Analysis of Partial Differential Operators with Irregular Singularities*

By Keisuke UCHIKOSHI
Department of Mathematics, University of Tokyo

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We denote the variables in $M=\boldsymbol{R}^{n+1}$ by $x=(x_0, x')$, where $x_0 \in \boldsymbol{R}$ and $x' \in \boldsymbol{R}^n$. We investigate partial differential operators of the form

$$P(x, \partial/\partial x)=\sum_{|\alpha|\leqslant m} a_\alpha(x) x_0^{\kappa(|\alpha|)}(\partial/\partial x)^\alpha$$

microlocally at $\mathring{x}^*=(0; \sqrt{-1}, 0, \cdots, 0) \in \sqrt{-1}T^*\boldsymbol{R}^{n+1}$. Here $a_\alpha(x)$, $|\alpha| \leqslant m$, are real analytic in a neighborhood of $x=0$, $a_{(m,0,\cdots,0)}=1$, and $\kappa(j)$, $0 \leqslant j \leqslant m$, are some integers $\geqslant 0$.

Definition 1. After Aoki [3], we define the *irregularity* σ of $P(x, \partial/\partial x)$ by

$$\sigma=\max\left\{\max_{0\leqslant j\leqslant m-1}\left(\frac{\kappa(m)-\kappa(j)}{m-j}\right), 1\right\}.$$

If $\sigma=1$, Kashiwara and Oshima [5] called the above operator $P(x, \partial/\partial x)$ a partial differential operator with regular singularities along the hypersurface $N=\{x_0=0\}$. They proved, in this case, that the above operator $P(x, \partial/\partial x)$ is equivalent to the very simple operator

$$\begin{array}{cccc} x_0^{\kappa(m)}: & \mathcal{C}_M & \longrightarrow & \mathcal{C}_M, \\ & \cup & & \cup \\ & u & \longmapsto & x_0^{\kappa(m)}u \end{array}$$

microlocally at $\mathring{x}^*$.

Our purpose is to generalize this result to the case $\sigma>1$. If $\sigma>1$, we say that the above operator has irregular singularities along the hypersurface N.

Definition 2. Let $\sigma>1$. We denote by $\lambda_1, \cdots, \lambda_{\kappa(m)}$ the roots of the algebraic equation

$$\lambda^{\kappa(m)}+\sum_{\pi(P)} a_{(j,0,\cdots,0)}(0)\lambda^{\kappa(j)}=0,$$

where

$$\pi(P)=\left\{0\leqslant j\leqslant m-1; \frac{\kappa(m)-\kappa(j)}{m-j}=\sigma\right\}.$$

We call these constants the *characteristic exponents* of P.

We investigate such a type of operators by means of holomorphic microlocal operators, due to Sato, Kawai and Kashiwara [7] and Aoki [2]. Now we have the following

Theorem 1. *Assume that* $\sigma>1$ *and that*

$$\lambda_i \neq \lambda_j \qquad \textit{if } i \neq j.$$

Then there exist holomorphic microlocal operators $Q_1(x,D),\cdots,Q_{\kappa(m)}(x,D)$ *such that the sequence*

$$0\longrightarrow\bigoplus_{\kappa(m)}\delta(x_0)\otimes\mathcal{A}_N\xrightarrow{(Q_1,\cdots,Q_{\kappa(m)})}\mathcal{C}_M\xrightarrow{P}\mathcal{C}_M\longrightarrow 0$$

is exact on a neighborhood of $\mathring{x}^*$, *in the sense of sheaf theory.*

Here we denoted by $\mathcal{A}_N$ (resp. $\mathcal{C}_M$) the sheaf of real analytic functions on N (resp. microfunctions on M). The above theorem asserts that $P(x,\partial/\partial x)$ is equivalent to the operator $x_0^{\kappa(m)}$ at $\mathring{x}^*$. (Compare the above exact sequence with the following one:

$$0\longrightarrow\bigoplus_{j=0}^{\kappa(m)-1}\delta^{(j)}(x_0)\otimes\mathcal{A}_N\hookrightarrow\mathcal{C}_M\xrightarrow{x_0^{\kappa(m)}}\mathcal{C}_M\longrightarrow 0.)$$

Remark. Such a result has been known only for the case of ordinary differential operators. (See Aoki [1] and Kashiwara [4].)

This type of partial differential operators was investigated also by Nourrigat [6], in the category of distribution theory. He proved that under certain conditions such a type of operators is C^∞-hypoelliptic, i.e., if $u\in\mathcal{D}'$ and $Pu\in C^\infty$, then $u\in C^\infty$. However we stress the fact that such operators behave completely differently in hyperfunction theory. In fact, Theorem 1 asserts that there exists a microfunction $u\neq 0$ such that $Pu=0$ as a microfunction. Such a microfunction can not be represented as a class of a distribution. More precisely, we can prove that if $s<\sigma/(\sigma-1)$, then u can be represented by an ultradistribution of class $\{s\}$, but if $s\geqslant\sigma/(\sigma-1)$, it cannot be in general.

To prove the above theorem, we need to consider a $\kappa(m)\times\kappa(m)$ matrix $x_0I_{\kappa(m)}+A(x',D)$ of microdifferential operators of fractional order satisfying the following conditions: There are two integers p and q relatively prime, and $1\leqslant p<q$. The symbol $\sigma(A)(x',\xi)$ of $A(x',D)$ admits an asymptotic expansion

$$\sigma(A)(x',\xi)\sim\sum_{-p/q\geqslant j\in(1/q)\mathbf{Z}}A_j(x',\xi)$$

in the sense of Aoki [2]. Here each (μ,ν) element $A_{j,(\mu,\nu)}(x',\xi)$ of $A_j(x',\xi)$ is a holomorphic function satisfying

$$|A_{j,(\mu,\nu)}(x',\xi)|<aR^j|\xi_0|^j[-j]!\quad -\frac{p}{q}\geqslant j\in\frac{1}{q}\mathbf{Z},\quad 1\leqslant\mu,\nu\leqslant\kappa(m)$$

with some constants $a,R>0$ on

$$\Gamma_\varepsilon=\{(x',\xi)=(x',\xi_0,\xi')\in\mathbf{C}^n\times\mathbf{C}\times\mathbf{C}^n;\ |x'|<\varepsilon,\ |\xi'|<\varepsilon|\xi_0|,\ \varepsilon|\xi_0|>1\text{ and }|\mathrm{Re}\,\xi_0|<\varepsilon\,\mathrm{Im}\,\xi_0\}.$$

Here $\varepsilon>0$ is some constant. (We refer the reader to Aoki [2] for the notion of microdifferential operators of fractional order.) Theorem 1 is a consequence of the following

Theorem 2. *Assume that all the eigenvalues of* $A_{-p/q}(x',\xi)$ *are distinct. Then there exist* $\kappa(m)\times\kappa(m)$ *matrices* $E(x',D)$ *and* $F(x',D)$ *of holomorphic microlocal operators defined at* $\mathring{x}^*$ *such that*

$$E(x', D)F(x', D)=F(x', D)E(x', D)=I_{\kappa(m)}$$

and

$$E(x', D)\{x_0 I_{\kappa(m)}+A(x', D)\}F(x', D)=x_0 I_{\kappa(m)}.$$

If $n=0$, Theorem 2 was proved also by Aoki [1] and Kashiwara [4]. Our proof is different from theirs. We followed the method developed by Turrittin [8], and this makes clear the asymptotic behavior of $\sigma(E)(x', \xi)$ and $\sigma(F)(x', \xi)$ as $|\xi|\to\infty$.

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