104. *Degeneration of the Two Dimensional Garnier System*

By Hironobu KIMURA
Department of Mathematics, University of Tokyo

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1. **Introduction.** In 1907, R. Fuchs [1] found out a remarkable connection between the sixth Painlevé equation and a second order linear ordinary differential equation

$$\frac{d^2y}{dx^2}=p(x;t)y. \tag{1.1}$$

He showed that the sixth Painlevé equation is just a deformation equation of a linear equation (1.1) of Fuchsian type, where t is a deformation parameter. The above result was extended by R. Garnier [2] in two directions. Firstly, he showed that the other five equations of Painlevé can be obtained from the isomonodromic deformation of linear equations with irregular singular points of the form (1.1). Secondly, he derived a completely integrable system of nonlinear Pfaffian equations from the isomonodromic deformation of (1.1) with N deformation parameters $t=(t_1, \cdots, t_N)$. This system can be considered as a generalization of the sixth Painlevé equation. Recently K. Okamoto [3] obtained a Hamiltonian system by considering the isomonodromic deformation of the following equation of Fuchsian type with N deformation parameters $t=(t_1, \cdots, t_N)$

$$\frac{d^2y}{dx^2}+p_1(x;t)\frac{dy}{dx}+p_2(x;t)y=0, \tag{1.2}$$

and he showed that Garnier's system can be transformed into a Hamiltonian system. By *N dimensional Garnier system* we mean the Hamiltonian system obtained by K. Okamoto from the isomonodromic deformation of (1.2) and it will be denoted by G_N.

The purpose of this note is to derive Hamiltonian systems from the two dimensional Garnier system G_2 by making a process of step-by-step degeneration and to show that this process of degeneration is induced by a process of confluence of singularities of linear equation (1.2).

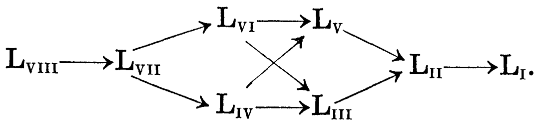
2. **Confluence of singularities.** We consider the equation (1.2) with $t=(t_1, t_2)$ and we start from the equation L_{VIII} given by

$$L_{VIII}\quad p_1(x;t)=\frac{1-\kappa_0}{x}+\frac{1-\kappa_1}{x-1}+\sum_j\frac{1-\theta_j}{x-t_j}-\sum_j\frac{1}{x-\lambda_j},$$

$$p_2(x;t)=\frac{\kappa}{x(x-1)}-\sum_j\frac{t_j(t_j-1)H_j}{x(x-1)(x-t_j)}+\sum_j\frac{\lambda_j(\lambda_j-1)\mu_j}{x(x-1)(x-\lambda_j)},$$

with $\kappa=((\kappa_0+\kappa_1+\theta_1+\theta_2-1)^2-\kappa_\infty^2)/4.$

Note that the isomonodromic deformation of L_{VIII} leads to the two dimensional Garnier system G_2. In order to determine the degeneration of G_2, we consider first the confluences of singularities of L_{VIII}, which give the following diagram.



These degenerations are carried out in three steps: (i) transformation of dependent and independent variables, (ii) transformation of parameters, (iii) makig ε tend to zero. The linear equations L_J, $J=\mathrm{I}, \cdots, \mathrm{VIII}$, are given as follows:

L_{VII}
$$p_1(x\,;t)=\frac{1-\kappa_0}{x}+\frac{\eta_1 t_1}{(x-1)^2}+\frac{2-\kappa_1}{x-1}+\frac{1-\theta_2}{x-t_2}-\sum_j\frac{1}{x-\lambda_j},$$
$$p_2(x\,;t)=\frac{\kappa}{x(x-1)}-\frac{t_1H_1}{x(x-1)^2}-\frac{t_2(t_2-1)H_2}{x(x-1)(x-t_2)}+\sum_j\frac{\lambda_j(\lambda_j-1)\mu_j}{x(x-1)(x-\lambda_j)},$$
with $\kappa=((\kappa_0+\kappa_1+\theta_2-1)^2-\kappa_\infty^2)/4.$

L_{VI}
$$p_1(x\,;t)=\frac{1-\kappa_0}{x}-\frac{\eta_1 t_1^2}{(x-1)^3}-\frac{\eta_1 t_2}{(x-1)^2}+\frac{3-\kappa_1}{x-1}-\sum_j\frac{1}{x-\lambda_j}$$
$$p_2(x\,;t)=\frac{\kappa}{x(x-1)}-\frac{t_1H_1-(t_1^2-t_2)H_2}{x(x-1)^2}+\frac{t_1^2H_2}{x(x-1)^3}+\sum_j\frac{\lambda_j(\lambda_j-1)\mu_j}{x(x-1)(x-\lambda_j)},$$
with $\kappa=((\kappa_0+\kappa_1-1)^2-\kappa_\infty^2)/4.$

L_{V}
$$p_1(x\,;t)=\frac{1-\kappa_0}{x}-x^2-t_2x-t_1-\sum\frac{1}{x-\lambda_j},$$
$$p_2(x\,;t)=\kappa_\infty x-H_1-\frac{2H_2+t_2H_1}{x}+\sum_j\frac{\lambda_j\mu_j}{x(x-\lambda_j)}.$$

L_{IV}
$$p_1(x\,;t)=\frac{\eta_0 t_2}{x^2}+\frac{2-\kappa_0}{x}+\frac{\eta_1 t_1}{(x-1)^2}+\frac{2-\kappa_1}{x-1}-\sum_j\frac{1}{x-\lambda_j},$$
$$p_2(x\,;t)=\frac{\kappa}{x(x-1)}-\frac{t_1H_1}{x(x-1)^2}+\frac{t_2H_2}{x^2(x-1)}+\sum_j\frac{\lambda_j(\lambda_j-1)\mu_j}{x(x-1)(x-\lambda_j)},$$
with $\kappa=((\kappa_0+\kappa_1-1)^2-\kappa_\infty^2)/4.$

L_{III}
$$p_1(x\,;t)=\frac{\eta_0 t_2}{x^2}+\frac{2-\kappa_0}{x}-x/2-t_1-\sum_j\frac{1}{x-\lambda_j},$$
$$p_2(x\,;t)=\kappa_\infty/2-\frac{H_1}{2x}-\frac{t_2H_2}{x^2}+\sum_j\frac{\lambda_j\mu_j}{x(x-\lambda_j)}.$$

$\mathrm{L_{II}}$ $$p_1(x\,;\,t)=-2x^3-2t_1x-t_2-\sum_j\frac{1}{x-\lambda_j},$$

$$p_2(x\,;\,t)=-(2\alpha+1)x^2-2H_2x-2H_1+\sum_j\frac{\mu_j}{x-\lambda_j}.$$

$\mathrm{L_{I}}$ $$p_1(x\,;\,t)=-\sum_j\frac{1}{x-\lambda_j},$$

$$p_2(x\,;\,t)=-9x^5-9t_1x^3-3t_2x^2-3H_2x-3H_1+\sum_j\frac{\mu_j}{x-\lambda_j}.$$

For example, the process of confluence of singularities $\mathrm{L_{VIII}}\rightarrow\mathrm{L_{VII}}$ are carried out by the transformation: $t_1\rightarrow 1+\varepsilon t_1$, $\kappa_1\rightarrow\eta_1/\varepsilon+\kappa_1$, $\theta_1\rightarrow-\eta_1/\varepsilon$, $H_1\rightarrow H_1/\varepsilon$ and by making ε tend to zero. In the other cases, the transformations are determined in a similar manner.

3. Isomonodromic deformation of $\mathrm{L_J}$. In order to determine the degeneration of $\mathrm{G_2}$ and to clarify the relation between the degeneration of $\mathrm{G_2}$ and the confluence of singularities of $\mathrm{L_J}$, we consider the isomonodromic deformation of linear equation $\mathrm{L_J}$. We assume that $x=\lambda_j$ $(j=1,2)$ are nonlogarithmic singular points of $\mathrm{L_J}$. Then, for each $\mathrm{L_J}$, H_1 and H_2 are determined as rational functions of $t,\lambda_1,\lambda_2,\mu_1$ and μ_2, which are denoted by $H_{J,1}, H_{J,2}$ respectively. Viewing t as deformation parameters, we can state the main result as follows.

Theorem. *Under the above assumption, the isomonodromic deformation of* $\mathrm{L_J}$ *is governed by the completely integrable Hamiltonian system*

$\mathrm{A_J}$ $$\partial\lambda_i/\partial t_k=\partial H_{J,k}/\partial\mu_i,\quad \partial\mu_i/\partial t_k=-\partial H_{J,k}/\partial\lambda_i.\quad (i,k=1,2).$$

Here, we shall give explicit expressions of the Hamiltonians for $\mathrm{A_{VIII}}$ and $\mathrm{A_{II}}$.

$\mathrm{A_{VIII}}$ $$H_{\mathrm{VIII},j}=\frac{\Lambda(t_j)}{T'(t_j)}\sum_i\frac{T(\lambda_i)}{(t_j-\lambda_i)\Lambda'(\lambda_i)}\left(\mu_i^2+\mu_i\left(-\frac{\kappa_0}{\lambda_i}-\frac{\kappa_1}{\lambda_i-1}+\sum_s\frac{\delta_s^j-\theta_s}{\lambda_i-t_s}\right)+\frac{\kappa}{\lambda_i(\lambda_i-1)}\right),$$

with $T(x)=x(x-1)(x-t_1)(x-t_2)$,
$\Lambda(x)=(x-\lambda_1)(x-\lambda_2)$.

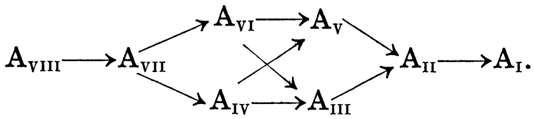
$\mathrm{A_{II}}$ $$H_{\mathrm{II},1}=-\frac{1}{2}\sum_i\frac{P(\lambda_i)}{\Lambda'(\lambda_i)}(\mu_i^2+\mu_i(-2\lambda_i^3-2t_1\lambda_i-t_2+P(\lambda_i)^{-1})-(2\alpha+1)\lambda_i^2),$$

$$H_{\mathrm{II},2}=\frac{1}{2}\sum_i\frac{1}{\Lambda'(\lambda_i)}(\mu_i^2+\mu_i(-2\lambda_i^3-2t_1\lambda_i-t_2)-(2\alpha+1)\lambda_i^2),$$

with $P(x)=-x+\lambda_1+\lambda_2$.

Here, $F'(x)$ means the derivative of $F(x)$ with respect to x.

In accordance with the process of confluence of singularities of linear equations $\mathrm{L_J}$, $\mathrm{J}=\mathrm{I},\cdots,\mathrm{VIII}$, there is a similar scheme of degeneration for $\mathrm{A_J}$. This scheme of degeneration is written in the following diagram:



This process of step-by-step degenerations is carried out by the use of the same transformation as for the confluence of singularities and by the use of succeeding limitation. Finally we remark that these transformations are canonical ones.

References

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[3] K. Okamoto: Isomonodromic deformation and Painlevé equations, and the Garnier system. I. R. M. A., Strasbourg (1981) (preprint).