92. *Characteristic Boundary Value Problems for Hyperbolic Equations*

By Kôichi UCHIYAMA
Department of Mathematics, Sophia University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1981)

§ 1. **Problems and results.** We study a priori estimates of the solution u of boundary value problems in the half space:

$$(1.1)\qquad \begin{cases} P(D_x, D_z)u(x, z)=f(x, z) & \text{for } x>0,\ z\in \boldsymbol{R}^n, \\ B_j(D_x, D_z)u|_{x=0}=g_j(z) & \text{for } z\in \boldsymbol{R}^n,\ j=0, 1, \cdots, \mu^+-1. \end{cases}$$

Let $(x, z)=(x, y, t)$ denote the variables in $\boldsymbol{R}_x\times\boldsymbol{R}_y^{n-1}\times\boldsymbol{R}_t$ and $(\xi, \zeta)=(\xi, \eta, \tau)$ denote the covariables corresponding to $(D_x, D_y, D_t)=(\partial/i\partial x, \partial/i\partial y, \partial/i\partial t)$. We assume

(P1) $P(D)=P(D_x, D_y, D_t)$ is a homogeneous differential operator of order m with constant coefficients and strictly hyperbolic with respect to D_t, and

(P2) the boundary $\{x=0\}$ is characteristic to P, i.e.

$$(1.2)\qquad P^0(1, 0, 0)=0,$$

where $P^0(\xi, \eta, \tau)$ is the principal symbol of $P(\xi, \eta, \tau)$.

There exists from (P1) a positive constant γ_0 such that

$$(1.3)\qquad P(\xi, \eta, \tau)\neq 0 \qquad \text{for } (\xi, \eta, \tau)\in\boldsymbol{R}^n\times\{\operatorname{Im}\tau<-\gamma_0\}.$$

Moreover, thanks to (P1) and (P2), we have an expression

$$(1.4)\qquad P(D_x, D_y, D_t)=P_{m-1}(D_z)D_x^{m-1}+P_{m-2}(D_z)D_x^{m-2}+\cdots+P_0(D_z),$$

where

$$(1.5)\qquad P_{m-1}(\eta, \tau)\neq 0 \qquad \text{for } (\eta, \tau)\in\boldsymbol{R}^{n-1}\times\{\operatorname{Im}\tau<-\gamma_0\}.$$

From (1.3) and (1.5), we have always $m-1$ non real roots of the characteristic equation $P(\xi, \zeta)=0$ for parameters $\zeta\in\boldsymbol{R}^{n-1}\times\{\operatorname{Im}\tau<-\gamma_0\}$. We denote them by $\xi_1^+(\zeta), \cdots, \xi_{\mu^+}^+(\zeta), \xi_1^-(\zeta), \cdots, \xi_{\mu^-}^-(\zeta)$, where $\pm$ mean the sign of imaginary parts. $\mu^++\mu^-=m-1$, and $\mu^\pm$ are independent on the parameter ζ.

We introduce boundary operators

$$(1.6)\qquad B_j(D_x, D_z)=\textstyle\sum_{k=0}^{m_j} B_{j,k}(D_z)D_x^k$$

of total order b_j $(m_j\leqq b_j)$. We assume

(B1) the number of boundary conditions is μ^+ i.e. $j=0, 1, \cdots, \mu^+-1$ in (1.6), and

(B2) $0\leqq m_j\leqq m-1$ i.e. $m_*=\max_j m_j<m-1$.

We define a polynomial of ξ with paramter ζ by

$$(1.7)\qquad \begin{aligned} P^+(\xi, \zeta)&=\textstyle\sum_{j=1}^{\mu^+}(\xi-\xi_j^+(\zeta)) \\ &=P_\mu^+(\zeta)\xi^{\mu^+}+\cdots+P_1^+(\zeta)\xi+P_0^+(\zeta). \end{aligned}$$

Dividing $B_j(\xi, \zeta)$ by $P^+(\xi, \zeta)$ as polynomials of ξ, we have the quotient

$S_j(\xi,\zeta)$ and the remainder term $B'_j(\xi,\zeta)$;

(1.8) $B_j(\xi,\zeta)=S_j(\xi,\zeta)P^+(\xi,\zeta)+B'_j(\xi,\zeta)$,
where $B'_j(\xi,\zeta)=\sum_{k=0}^{\mu^+}B'_{j,k}(\zeta)\xi^k$, $S_j(\xi,\zeta)=\sum_{k=0}^{m_j-\mu^+}S_{j,h}(\zeta)\xi^h$
for $m_j\geqq\mu^+$ and $S_j(\xi,\zeta)=0$ for $m_j<\mu^+$.

The Lopatinski-Shapiro matrix is defined by $B'(\zeta)=\{B'_{j,k}(\zeta);\ j,k=0,1,\cdots,\mu^+-1\}$. Its determinant $R(\zeta)=\det B'(\zeta)$ is called the Lopatinski-Shapiro determinant. Let $A(\zeta)=\{A_{j,k}(\zeta)\}$ be the inverse matrix of $B'(\zeta)$.

The Fourier-Laplace transform is defined by

$$\hat{u}(\xi,\eta,\sigma-i\gamma)=\widehat{e^{-\gamma t}u}(\xi,\eta,\sigma)\qquad\text{for real }\sigma\text{ and }\gamma>0.$$

The norms of the weighted Sobolev spaces $H_{s,t;\gamma}(\boldsymbol{R}^{n+1}_{x,z})$ are

$$|u|^2_{s,t;\gamma}=\iiint(|\xi|^2+|\eta|^2+\sigma^2+\gamma^2)^s(|\eta|^2+\sigma^2+\gamma^2)^t|\hat{u}(\xi,\eta,\sigma-i\gamma)|^2d\xi d\eta d\sigma$$

and those of $H_{s;\gamma}(\boldsymbol{R}^n_z)$ are

$$\langle v\rangle^2_{s;\gamma}=\iiint(|\eta|^2+\sigma^2+\gamma^2)^s|\hat{v}(\eta,\sigma-i\gamma)|^2d\eta d\sigma.$$

We introduce an index depending on the coefficients in (1.4). Dividing $P_j(\zeta)$ by $P_{m-1}(\zeta)$ as polynomials in τ, we have the remainder $P'_j(\eta)$. Let l be maximal number of $\{j'\}$ such that $\deg_\eta P'_{m-j}(\eta)<j$ for $j=1,2,\cdots,j'$ $(1\leqq j'\leqq m)$. Let $\nu=1/l$, if $1\leqq l\leqq m-1$ and $\nu=0$, if $l=m$.

Main results. *Assume* (P1), (P2), (B1), (B2) *and*

$$(\mathrm{L}_\theta)\qquad\begin{cases}\text{there exist }\theta\geqq0,\ \gamma_1>0\text{ and }C>0\text{ such that}\\ |A_{k,j}(\zeta)|\leqq\dfrac{C|\zeta|^{k-b_j+\theta}}{|\mathrm{Im}\,\tau|^\theta}\ \text{for all }\zeta=(\eta,\tau)\in\boldsymbol{R}^{n-1}\times\{\mathrm{Im}\,\tau<-\gamma_1\}.\end{cases}$$

Then, the unique solution u of (1.1) *satisfies one of the following inequalities:*

if $\mu^+=m-1$,

$$(1.9)\qquad \gamma|u|^2_{m-1;\gamma}+\sum_{k=0}^{m-2}\langle\gamma_k u\rangle^2_{m-1-k;\gamma}\leqq C\Big(\frac{1}{\gamma}|f|^2_{0;\gamma}+\frac{1}{\gamma^{2\theta}}\sum_{j=0}^{\mu^+-1}\langle g_j\rangle^2_{m-1-b_j+\theta;\gamma}\Big),$$

if $1\leqq\mu^+\leqq m-2$,

$$(1.10)\qquad \gamma^{1+2\alpha[m-2]}|u|^2_{m-1,-\alpha[m-2];\gamma}+\sum_{k=0}^{\mu^+-1}\langle\gamma_k u\rangle^2_{m-1-k;\gamma}+\sum_{k=\mu^+}^{m-2}\gamma^{2\alpha[k]}\langle\gamma_k u\rangle^2_{m-1-k-\alpha[k];\gamma}\leqq\frac{C}{\gamma^{2\theta}}\Big\{\frac{|f|^2_{0,\theta+m_*+1;\gamma}}{\gamma^{1+2(m_*+1)}}\sum_{j=0}^{\mu^+-1}\langle g_j\rangle^2_{m-1-b_j+\theta;\gamma}\Big\}$$

where $\alpha[k]=(k-\mu^+)\nu+\min\{\mu^+\nu,1\}$.

Remark. In the case of non characteristic boundary, the condition (L_θ) gives

$$(1.11)\qquad \gamma|u|^2_{m,-1;\gamma}+\sum_{k=0}^{m-1}\langle\gamma_k u\rangle^2_{m-1-k;\gamma}\leqq\frac{C}{\gamma^{2\theta}}\Big\{\frac{|f|^2_{0;\gamma}}{\gamma}+\sum_{j=0}^{\mu^+-1}\langle g_j\rangle^2_{m-1-b_j+\theta;\gamma}\Big\}.$$

(See [1], [5].) Comparing (1.10) with (1.11), we see losses of tangential regularity (cf. [3, p. 622, Th. 3]). Estimates without loss for symmetric hyperbolic systems of first order are studied by A. Majda and S. Osher [3] for the uniform Lopatinski-Shapiro condition (Kreiss condition) and by T. Ohkubo [4] for maximal non positive conditions.

Examples of $P(D)$. (i) (trivial one).

$$P(D)=a_1D_{y_1}+\cdots+a_{n-1}D_{y_{n-1}}+D_t, \qquad \text{where } a_i\in \boldsymbol{R}.$$

Then, $\mu^+=\mu^-=0$.

(ii) $P(D)=D_t^2+2(D_tD_y+D_yD_x+D_xD_t)$. Then, $\mu^+=1$, $\mu^-=0$. $\nu=1$. $\xi^+(\zeta)=(-\tau^2-2\tau\eta)/2(\eta+\tau)$.

(iii) $P(D)=D_t(D_t^2-D_y^2-D_x^2)$. Then, $\mu^+=\mu^-=1$, $\nu=0$, $\xi^\pm(\zeta)=\pm{}^+\sqrt{\tau^2-\eta^2}$ and $|\xi^\pm(\zeta)|=O(|\zeta|)$.

(iv) $P(D)=D_t(D_t^2-D_y^2-D_x^2)+(2/3)D_y^2D_x$. Then, $\mu^+=\mu^-=1$, $\nu=1$, $\xi^\pm(\zeta)=\{\eta^2/3\pm{}^+\sqrt{\eta^4/9+\tau^2(\tau^2-\eta^2)}\}/\tau$, $|\xi^+(\zeta)|=O(|\zeta|^2/|\mathrm{Im}\,\tau|)$ and $|\xi^-(\zeta)|=O(|\zeta|)$.

(v) $P(D)=D_t(D_t^2-D_y^2-D_x^2)+(1/3)D_y^3$. Then, $\mu^+=\mu^-=1$, $\nu=1/2$, $\xi^\pm(\zeta)=\pm{}^+\sqrt{-\eta^2+\tau^2+\eta^3/3\tau}$, $|\xi^\pm(\zeta)|=O(|\zeta|^{1+1/2}|\mathrm{Im}\,\tau|^{-1/2})$.

§2. Sketch of the proofs. Since $P_{m-1}(\zeta)\neq 0$ for ζ in $\boldsymbol{R}^{n-1}\times\{\mathrm{Im}\,\tau<-\gamma_0\}$, the defining equation of characteristic roots $\xi_j(\zeta)$ is

$$\xi^{m-1}+\frac{P_{m-2}(\zeta)}{P_{m-1}(\zeta)}\xi^{m-2}+\cdots+\frac{P_0(\zeta)}{P_{m-1}(\zeta)}=0. \tag{2.1}$$

Main difficulties arise from the fact that the characteristic roots may be singular, when ζ tends to real zeros of $P^0_{m-1}(\zeta)$. The number of singular roots is equal to l introduced in §1 and depends in general on η. We have, however, the following estimates.

Lemma 1. (i) *There exist $\gamma>0$ and $C_\gamma>0$ such that*

$$|\xi_j(\zeta)|\leqq\frac{C_\gamma}{|\mathrm{Im}\,\tau|^\nu}|\zeta|^{1+\nu} \qquad \textit{for } \zeta\in\boldsymbol{R}^{n-1}\times\{\mathrm{Im}\,\tau<-\gamma\} \tag{2.2}$$

$$(j=1,2,\cdots,m-1).$$

(ii) *If* $1\leqq i_1<i_2<\cdots<i_j\leqq m-1$,

$$|\xi_{i_1}(\zeta)\cdots\xi_{i_j}(\zeta)|\leqq\frac{C_\gamma|\zeta|^{j+\min\{j\nu,1\}}}{|\mathrm{Im}\,\tau|^{\min\{j\nu,1\}}} \qquad \textit{for } \zeta \textit{ in (i)}. \tag{2.3}$$

(iii) *If* $m_j\geqq\mu^+$,

$$|S_{j,h}(\zeta)|\leqq\frac{C_\gamma|\zeta|^{b_j-\mu^+-h+(m_j-\mu^+-h)\nu}}{|\mathrm{Im}\,\tau|^{(m_j-\mu^+-h)\nu}} \qquad \textit{for } \zeta \textit{ in (i)} \tag{2.4}$$

$$(h=0,1,\cdots,m_j-\mu^+).$$

Proof. (i) We have only to transform (2.1) by $u=|\mathrm{Im}\,\tau|^\nu|\xi|^{-1-\nu}$ in order to obtain an algebraic equation of u with bounded coefficients.

(ii) If $j\nu\leqq 1$, (2.3) follows immediately from (2.2). Let $j\nu>1$. We define index sets $\mathcal{J}^{(j)}=\{I=(i_1,\cdots,i_j)\in \boldsymbol{N}^j;\ 1\leqq i_1<\cdots<i_j\leqq m-1\}$ equipped with the lexicographic order. Abbreviating $(-1)^j\xi_{i_1}(\zeta)\cdots\xi_{i_j}(\zeta)$ to $\Xi_I(\zeta)$, we introduce a polynomial of one variable x:

$$R(x\,;\,\zeta)=\prod_{I\in\mathcal{J}^{(j)}}(x-\Xi_I(\zeta))$$
$$=x^{\binom{m-1}{j}}+r_{\binom{m-1}{j}-1}(\zeta)x^{\binom{m-1}{j}-1}+\cdots+r_0(\zeta).$$

Since $r_{\binom{m-1}{j}-l}(\zeta)$ is a homogeneous symmetric polynomial of $\xi_1(\zeta),\cdots,$ of order lj, it is written by the fundamental symmetric polynomials $A_1,\cdots,A_{m-1}$ of $(\xi_1,\cdots,\xi_{m-1})$. More precisely, there exist quasi-homogeneous polynomials $G_{j,l}(A_1,\cdots,A_{m-1})=\sum_\alpha c_\alpha A_1^{\alpha_1}\cdots A_{m-1}^{\alpha_{m-1}}$ such that

$$r_{\binom{m-1}{j}-l}(\zeta)=G_{j,l}(A_1(\zeta),\,\cdots,\,A_{m-1}(\zeta)),$$
$$\alpha_1+\cdots+\alpha_{m-1}\leqq l \text{ and } \alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}=lj.$$

Therefore, $|r_{\binom{m-1}{j}-l}(\zeta)|\leqq C|\zeta|^{lj}(|\zeta|/|\mathrm{Im}\,\tau|)^l$. On the other hand, all the roots of an algebraic equation $x^p+a_1x^{p-1}+\cdots+a_p=0$ have a majorant $\max\{|pa_1|,|pa_2|^{1/2},\cdots,|pa_p|^{1/p}\}$. Hence, $\Xi_I(\zeta)$ has a majorant $C|\zeta|^j(|\zeta|/|\mathrm{Im}\,\tau|)$.

(iii) (2.4) is shown by the mathematical induction. Q.E.D.

Main results depend on the following two propositions.

Proposition 2. *Under the assumptions* (P1) *and* (P2), *there exist* $\gamma_2>0$ *and* $C>0$ *such that*

$$(2.5)\qquad \gamma|u|^2_{m-1;\gamma}\leqq C\Big(\frac{1}{\gamma}|Pu|^2_{0;\gamma}+\sum_{k=0}^{m-2}\langle\gamma_k u\rangle^2_{m-1-k;\gamma}\Big)\qquad \textit{for all } \gamma\geqq\gamma_2.$$

Proposition 3. *When* $\mu^+\leqq m-2$, *we have*

$$\int_0^\infty\left|\int_{C^-(\zeta)}\frac{e^{-ix.\xi\xi h}}{P^-(\xi,\zeta)}d\xi\right|^2dx\leqq\frac{C_\gamma|\zeta|^{2\{-m+\mu^++1+(\mu+\nu+1)\}}}{|\mathrm{Im}\,\tau|^{1+2(\mu+\nu+1)}}$$

for $\zeta\in\boldsymbol{R}^{n-1}\times\{\mathrm{Im}\,\tau<-\gamma_2\}$, *where* $P^-(\xi,\zeta)=P(\xi,\zeta)/P^+(\xi,\zeta)$ *and* $C^-(\zeta)$ *denotes a closed curve in the lower half plane* $\{\mathrm{Im}\,\xi<0\}$ *surrounding* $\xi_1^-(\zeta),\,\cdots,\,\xi_{\mu^-}^-(\zeta)$.

To prove Proposition 2, we use a partially detailed version of Lemma 8.2.1 in [2] to exclude the term $\langle\gamma_{m-1}u\rangle_0^2$ from the right hand side of (2.5).

The guidelines for the proof of the main estimates are the same as they were in [5], [1], though calculations are more complicated.

Remark. In the main estimates, the loss of tangential regularity due to Lemma 1 is inevitable. But, it seems that the loss due to Proposition 3 should be improved ([6]).

The author wishes to express his gratitude to Profs. K. Kubota and S. Wakabayashi for their advices on the estimate of the characteristic roots in the early stage of this study.

References

[1] J. Chazarain et A. Piriou: Caractérisation des problèmes mixtes hyperboliques bien posés. Ann. Inst. Fourier, **22**, 193–237 (1972).
[2] L. Hörmander: Linear Partial Differential Operators. Springer (1967).

[3] A. Majda and S. Osher: Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary. Comm. Pure. Appl. Math., **28**, 607–675 (1975).
[4] T. Ohkubo: Regularity of solutions to hyperbolic mixed problems with uniformly characteristic boundary. Hokkaido Math. J., **10**, 93–123 (1980).
[5] K. Uchiyama: Caractérisation des problèmes mixtes bien posés pour des opérateurs evolutifs à coefficients constants. Tokyo J. Math., **1**, 369–388 (1978).
[6] ——: On characteristic boundary value problems for hyperbolic equations. Kôkyûroku RIMS, no. 431, pp. 150–163 (1981) (to appear) (in Japanese).