The \( \tau \) Function of the Kadomtsev-Petviashvili Equation

Transformation Groups for Soliton Equations. 1

By Masaki Kashiwara and Tetsuji Miwa

Research Institute for Mathematical Sciences, Kyoto University

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The notion of \( \tau \) function was first introduced by Sato, Miwa and Jimbo in a series of papers on holonomic quantum fields \([1]\). There the \( \tau \) functions were simply expressed as the expectation values of field operators which belong to the Clifford group of free fermions. Then by several authors \([2]-[4]\) the concept was generalized and exploited in the study of monodromy and spectrum preserving deformations. They also showed that \( \tau \) functions are nothing other than the dependent variables used by Hirota in this theory of bilinear equations \([5]\).

This is the first in a series of papers \([6], [7]\) by the present authors, E. Date and M. Jimbo, which aims at a further study of \( \tau \) functions and soliton equations.

The main results in the present paper are the following. a) We construct a Clifford operator \( \varphi(x) \) so that for any even Clifford group element \( g \) the expectation value \( \tau(x) = \langle \varphi(x) g \rangle \) gives us a solution to the hierarchy of the KP (Kadomtsev-Petviashvili) equations in Hirota's bilinear form. b) Define polynomials \( p_j(x) \) \((j=0, 1, 2, \cdots)\) by

\[
\exp \left( \sum_{j=1}^{\infty} k^j x_j \right) = \sum_{j=0}^{\infty} p_j(x) k^j.
\]

The KP hierarchy contains the following infinite number of bilinear equations:

\[
\begin{vmatrix}
    p_{f_1+1} \left( -\frac{\bar{D}}{2} \right) & p_{f_1+1} \left( \frac{\bar{D}}{2} \right) & \cdots & p_{f_1+m-1} \left( \frac{\bar{D}}{2} \right) \\
    p_{f_2} \left( -\frac{\bar{D}}{2} \right) & p_{f_2} \left( \frac{\bar{D}}{2} \right) & \cdots & p_{f_2+m-1} \left( \frac{\bar{D}}{2} \right) \\
    \vdots & \vdots & \ddots & \vdots \\
    p_{f_m-m+2} \left( -\frac{\bar{D}}{2} \right) & p_{f_m-m+2} \left( \frac{\bar{D}}{2} \right) & \cdots & p_{f_m} \left( \frac{\bar{D}}{2} \right)
\end{vmatrix} \tau(x) \cdot \tau(x) = 0,
\]

\((f_1 \geq f_2 \geq \cdots \geq f_m \geq 1, m \geq 2, \bar{D} = (D_1, D_2/2, D_3/3, \cdots)). \)

Our work is deeply related to the recent progress \([8]\) by M. Sato and Y. Sato on the structure of \( \tau \) functions for the KP hierarchy. In fact, our starting point was Sato's lecture \([9]\) in which he claimed...
that the space of solutions to the KP hierarchy is neatly parametrized by the Grassmann manifold of infinite dimensions.

In § 1 we give the definition of the \( \tau \) function for the KP hierarchy. This part is due to an unpublished work of M. Jimbo, M. Sato and one of the present authors (T. M.). In § 2 we give an expression for the \( \tau \) functions of the KP hierarchy by means of the Clifford operators used in [10]. In § 3 we give the explicit forms of Hirota’s bilinear equations for them using a transformation introduced in [8]. In § 4, as a corollary of the operator expression given in § 2, we show that the characters of the general linear group are \( \tau \) functions for the KP hierarchy; that is one of the astonishing results of M. and Y. Sato [9].

We thank M. Sato for his explanation on his recent work with Y. Sato before publication. We benefited from discussions with M. Jimbo.

§ 1. We denote by \( x=(x_1, x_2, \cdots) \) an infinite number of independent variables. The hierarchy of higher-order KP equations is a system of non-linear partial differential equations with unknown functions \( b_{i,m}(x) \) \( (l=2, 3, \cdots; 0 \leq m \leq l-2) \) obtained as the compatibility condition for the following system of linear partial differential equations for a wave function \( w(x) \).

\[
(1) \quad \frac{\partial w(x)}{\partial x_i} = \left( \frac{\partial^l}{\partial x_i^l} + \sum_{m=0}^{l-2} b_{i,m}(x) \frac{\partial^m}{\partial x_1^m} \right) w(x) \quad l=2, 3, \cdots.
\]

We set

\[
(2) \quad \xi(x, k) = \sum_{i=1}^\infty k^i x_i.
\]

Assume the system (1) is compatible. Then, by a suitable change of dependent and independent variables, we can find a formal solution \( w(x, k) \) to (1), containing a spectral parameter \( k \), of the form

\[
(3) \quad w(x, k) = \hat{w}(x, k) \exp \xi(x, k),
\]

\[
(4) \quad \log \hat{w}(x, k) = \sum_{i=1}^\infty t_i(x) k^{-i}.
\]

A formal solution is uniquely determined by (1) up to a factor depending on \( k \) but independent of \( x \). If we fix a formal solution, the \( \tau \) function is consistently defined up to a constant factor by the following:

\[
(5) \quad \frac{\partial}{\partial x_i} (\log \tau) = \hat{t}_i(x) - \sum_{m=1}^{l-1} \frac{\partial \hat{t}_{l-m}(x)}{\partial x_m}.
\]

We set \( \hat{t} = (\partial/\partial x_1, (1/2)(\partial/\partial x_2), (1/3)(\partial/\partial x_3), \cdots) \) and define a modified \( \tau \) function \( \tau_{t(x)}(x) \) by

\[
(6) \quad \tau_{t(x)}(x) = \exp \xi(x, k) e^{-\xi(x, k-1)} \tau(x) = \tau(x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, \cdots) e^{\xi(x, k)}.
\]

Then the system (5) is rewritten as a formal identity which directly connects \( w(x, k) \) with \( \tau(x) \);
Note that a change of $r(x)$ by the exponential of a linear function gives a multiplication to $w(x, k)$ by a function in $k$.

If $l=1$ in (5), we have

\[ b_{30}(x) = 2(\partial^2 / \partial x^2) \log \tau(x). \]

Thus the introduction of $\tau(x)$ by (5) gives a generalization of Hirota's dependent variable transformation [5] for the total hierarchy. Namely, if the system (1) is compatible, $\tau(x)$ satisfies infinitely many Hirota bilinear differential equations. Let us count the degree of $D_t$ as $l$.

Note that if $P(-D) = -P(D)$ we have $P(D)f(x) \cdot f(x) = 0$ for any function $f(x)$. The equation of the lowest degree among non trivial ones is the following [11].

\[ (D_t^3 + 3D_t^2 - 4D_tD_s)\tau(x) \cdot \tau(x) = 0. \]

In [8], M. Sato and Y. Sato listed the bilinear equations satisfied by $\tau(x)$ up to degree 11, and conjectured that the number of bilinear equations of degree $n$ is equal to $p(n-1)$, where $p(n)$ is the partition function; $p(n) = \#(n_1, \ldots, n_s)|n_s : \text{integer s.t. } 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ and $\sum_{s=1}^{s=n_s} n_s = n$.

In §3 we propose a refined version of their conjecture by giving explicit forms of bilinear equations for every degree.

2. Now we prepare some notations on characters of the general linear group. Following [9] we set $x_i = \text{trace } g^i / l$, where $g$ is an element of $GL(N, C)$ for a sufficiently large $N$. For any Young diagram $Y = (f_1, \ldots, f_m)$ we denote by $\chi_Y(x)$ its character evaluated at $g$. This is a polynomial in $x$ which is independent of the choice of $N$ larger than $f_1 + \cdots + f_m$. We denote by $p_i(x)$ the $i$-th symmetric tensor (i.e. $Y = (1^i)$ = $(1, \ldots, 1)$). For $l < 0$ we understand $p_i(x)$ and $q_i(x)$ to be zero then we have

\[ e^i(x,k) = \sum k^i p_i(x), \]

and $q_i(-x) = (-1)^i p_i(x)$.

Now we denote by $\psi_m$ and $\psi_m^*$ ($m \in Z$) free fermions such that $[\psi_m, \psi_n^*] = \delta_{m,n}$, $[\psi_m, \psi_n] = 0$ and $[\psi_m^*, \psi_n^*] = 0$. We also use their Fourier transforms $(dk = dk/2\pi ik)$:

\[ \psi(k) = \sum_{n \in Z} \psi_n k^n, \quad \psi^*(k) = \sum_{n \in Z} \psi_n^* k^{-n}, \]

We define the Hamiltonian for the $x$-flows by

\[ H(x) = \sum_{i \geq 1} x_i \psi^*_n \psi^*_{n+1} = \int dk \xi(x, k) \psi(k) \psi^*(k) \]

and set
We introduce a Clifford operator \( \varphi(x) = e^{\rho(x)} \) by
\[
\rho(x) = - \sum_{k=1}^{\infty} \frac{k}{k-1} \varphi_{-k}(-x) \varphi_k(-x),
\]
This operator is essentially the same as the operator used in [11].

**Theorem 1.** If \( g \) belongs to the Clifford group,
\[
\tau(x) = \langle \varphi(x)g \rangle
\]
satisfies the bilinear equations of the KP hierarchy. The \( l \)-th modified \( \tau \) function
\[
\tau_{l+1}(\tilde{D}) = e^{\rho(x)} e^{\varphi_{l+1}(-x)} \varphi_{-l}(x) \varphi_{l-1}(-x) \psi_{l+1}(x)
\]
satisfies the following \( l \)-th modified KP equation (the \( l \)-th modified KP equation):
\[
q_{l+1}(\tilde{D}) \tau_{l+1}(\tilde{D})(x) \cdot \tau(x) = 0.
\]

We shall give below a sketch of the proof of Theorem 1. First we note that
\[
\varphi_{l+1}(-x) \psi_{l-1}(x) = \dot{\psi}(x) \varphi_{l+1}(-x) \psi_{l-1}(x).
\]
Then the proof of (17) is done in the following manner. For simplicity, let us assume that \( l = 1 \). Noting that
\[
\chi(\tilde{D})f \cdot g = \langle \chi(\tilde{D})f \rangle g - \langle \chi(\tilde{D})g \rangle f
\]
we have
\[
\chi(\tilde{D}) \langle :e^{\rho(x)} \varphi_0(-x) : \varphi(k)g \rangle \cdot \langle :e^{\rho(x)} : g \rangle
\]
\[
= \langle :e^{\rho(x)} \varphi_{-1}(-x) \psi_0(-x) : \psi(k)g \rangle \cdot \langle :e^{\rho(x)} : g \rangle
\]
\[
- \langle :e^{\rho(x)} \varphi_{-1}(-x) : \psi(k)g \rangle \cdot \langle :e^{\rho(x)} \varphi_{-1}(-x) : g \rangle
\]
\[
+ \langle :e^{\rho(x)} \varphi_0(-x) : \psi(k)g \rangle \cdot \langle :e^{\rho(x)} \psi_0(-x) : g \rangle
\]
\[
= 0.
\]
In the last step we used Wick's theorem which is valid only if \( g \) belongs to the Clifford group. Noting (7) we can rewrite the bilinear equation (17) with \( l = 1 \) as the linear equation (1) with \( l = 2 \). The latter with \( l \geq 3 \) is obtained from (22) below with \( l' = 1 \).

**§ 3.** For any Young diagram we set
\[
\chi^{\mu}_{\nu}(\tilde{D}) = \det \left[ \begin{array}{c} p_{f_1+1}(\tilde{D}/2)p_{f_1+2}(\tilde{D}/2) \cdots p_{f_1+m-1}(\tilde{D}/2) \\ p_{f_{1+1}}(-\tilde{D}/2)p_{f_{1+2}}(\tilde{D}/2) \cdots p_{f_{1+m-2}}(\tilde{D}/2) \\ \vdots \\ p_{f_m+1-m-1}(\tilde{D}/2)p_{f_m+1}(\tilde{D}/2) \cdots p_{f_m}(\tilde{D}/2) \end{array} \right].
\]

Then we have

**Theorem 2.** The hierarchy of the \( M'KP \) \((l \geq 0)\) equations contains the bilinear equation
for each $Y$.

We conjecture that the $h^{l+1}_l(\tilde{D})$'s are linearly independent and exhaust all the bilinear equations of the M'KP hierarchy. Since the degree of $h^{l+1}_l(\tilde{D})$ is $f_1 + \cdots + f_m + l + 1$, the number of degree $n$ bilinear equations of the M'KP hierarchy is expected to be $p(n-l-1)$. The case $l=0$, i.e.

$$
\#\{\text{degree } n \text{ Hirota's bilinear equation of the KP hierarchy}\} = p(n-1),
$$

was conjectured by M. and Y. Sato in [8]. The proof of (21) will be given in one of our forthcoming papers [7], in which we discuss the relation between soliton equations and the Euclidean Lie algebras.

Now we shall give a sketch of the proof of Theorem 2. First we note the following [8].

**Lemma.** If $f(x)$ and $g(x)$ satisfy a bilinear equation $Q(D)f(x)\cdot g(x)=0$, then $f(x)$ and $g(x)$ satisfy

$$
Q(D + k_1, D + k_2, \cdots) \exp \left( -\frac{1}{2} \left( \sum_{i=1}^m \frac{1}{k_i} \right) D_1 + \frac{1}{2} \sum_{i=1}^m \frac{k_i^2}{k_i} \right) f(x) g(x) = 0.
$$

Applying this lemma repeatedly to (17) we find that

$$
Q(D^{l+1}, \cdots, D^{l+1}) f(x) g(x) = 0
$$

where

$$
Q(D; k_{l+1}, \cdots, k_l) = \frac{1}{k_{l+1} \cdots k_l} q_{l+1} \left( \tilde{D}_1 + k_{l+1} + \cdots + k_l, \right.
$$

$$
\tilde{D}_1 + \frac{k_{l+1}^2}{2} + \cdots + \frac{k_l^2}{2} \left( \sum_{i=1}^m \frac{1}{k_i} \right) \tilde{D}_1 + \left( \sum_{i=1}^m \frac{1}{k_i} \right) \tilde{D}_1 + \cdots \right).
$$

Setting $y = \left( k + \cdots + k_l \right)/2$ and taking the limit $l \to \infty$ we obtain

$$
\sum_{j=0}^\infty p_j(2y) p_j(\tilde{D}_1) \exp \left( \sum_{j=1}^\infty y_j D_j \right) f(x) g(x) = 0.
$$

Using Weyl's character formula we find the coefficient of $\chi^*_1(y)$ in (22) to be $h^{l+1}_l(\tilde{D})$.

§ 4. In order to compute $\langle \varphi(x) g \rangle$ explicitly we fix the expectation value. The following choice of the vacuum was suggested by M. Sato.

$$
\psi_n | vac > = 0 \quad n \leq -1, \quad < vac | \psi_n = 0 \quad n \geq 0,
$$

$$
\psi^*_n | vac > = 0 \quad n \geq 0, \quad < vac | \psi^*_n = 0 \quad n \leq -1.
$$

We set

$$
p_{\mu}(x) = \sum_{l \geq -1} p_{l+1, -(x)} = \langle \psi^*_\mu(x) \psi^*_\nu (x) \rangle \quad (\mu, \nu \in Z).
$$

Then we have

$$
\rho(x + y) = \sum_{\mu, \nu \in Z} \psi_{\mu}(x) \psi^*_{\nu}(-x) p_{\mu}(y).
$$

If $\mu \leq -1$ and $\nu \geq 0$, $(-)^{\nu+1} p_{\mu}(x)$ is the character for $Y = (\nu+1, 1^{l-\nu})$ and
\[ \langle \psi_j^\tau(-x)\psi_\nu \rangle = \langle \psi_j(x)\psi_\nu^\tau \rangle = p_{-\mu_1}(x) \] Now let \( Y \) be a Young diagram of the following form:

\[ Y = -\mu_1 \]

\[ -\mu_m + 1 \]

The character \( \chi_\nu(x) \) is written as

\[ (-)^{\mu_1 + \cdots + \mu_m} \chi_\nu(x) = \langle \phi(x) \psi_{\mu_1}^\tau \cdots \psi_{\mu_m}^\tau \psi_{\nu_1} \cdots \psi_{\nu_n} \rangle = (-)^n \det (p_{-\mu_j}(x))_{j,k=1,\ldots,m} \]


p. 405, l. 9 from bottom: For "(\sinh \beta_1 E_1 \sinh \beta_2 E_2)^2 = (\sinh \beta_1 E_1 \sinh \beta_2 E_2)^{-2}" read "(\sinh 2\beta_1 E_1 \sinh 2\beta_2 E_2)^2 = (\sinh 2\beta_1 E_1 \sinh 2\beta_2 E_2)^{-2}.

1. 2 from bottom: For

\[ N^2(t-1) \frac{d\sigma}{dt} - \frac{1}{2} \left( \left( t-1 \frac{d\sigma}{dt} - \sigma \right) \right) \]

read

\[ N^2(t-1) \frac{d\sigma}{dt} - \frac{1}{4} \left( \left( t-1 \frac{d\sigma}{dt} - \sigma \right) \right)^2 \]

p. 406, l. 20: For "\( \beta_0^{-1} = \alpha_0 \sqrt{t-1} F(-1/2, 1/2, 1; 1/t) - F(1/2, 1/2, 1; 1/t) \)" read "\( \beta_0^{-1} = -\alpha_0 \sqrt{t-1} \sqrt{t-1} F(-1/2, 1/2, 1; 1/t) - F(1/2, 1/2, 1; 1/t) \)".

1. 11 from bottom: For "(\sinh \beta_1 E_1 \sinh \beta_2 E_2)^{-1}" read "(\sinh 2\beta_1 E_1 \sinh 2\beta_2 E_2)^{-1}.

p. 407, l. 2: For "(\sinh \beta_1 E_1 \sinh \beta_2 E_2)^{-1} read "(\sinh 2\beta_1 E_1 \sinh 2\beta_2 E_2)^{-1}.

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References