54. Eta-Function on S^3

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Let M be a compact oriented riemannian manifold of dimension $2q-1$, $\bigwedge^p(M)$ be the vector space over $\boldsymbol{R}$ of all differential p-forms on M and put $\bigwedge^{ev}(M)=\sum_{i=0}^{q-1}\bigwedge^{2i}(M)$. Let $A\colon \bigwedge^{ev}(M)\to\bigwedge^{ev}(M)$ be a self-adjoint elliptic first order differential operator defined by

$$A\phi=(\sqrt{-1})^q(-1)^{p+1}(*d-d*)\phi$$

where the degree of ϕ is equal to $2p$, d is the exterior differential and $*$ is the Hodge duality operator. Note that the square $A^2=A\circ A$ is the Laplace operator $\Delta=d\delta+\delta d$, where δ is the codifferential of d.

When a compact group G acts on M, M. F. Atiyah, V. K. Patodi and I. M. Singer [1] defined a function

$$\eta_A(g,s)=\sum_{\lambda\neq 0}(\operatorname{sign}\lambda)Tr(g\,|\,E_\lambda)\cdot|\lambda|^{-s}$$

for g in G, where the summation is taken over all distinct eigenvalues of A and $g\,|\,E_\lambda$ is the transformation induced by g on the λ-eigenspace E_λ. They also showed as an example that if M is the circle S^1 and g is rotation through an angle α, then

$$\eta_A(g,s)=2\sqrt{-1}\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n^s}.$$

But it seems to be very hard to obtain directly this function for other manifolds. J. J. Millson [6] showed a method to calculate this eta-function on homogeneous spaces by using the Selberg zeta-function.

In this paper we show how to calculate more elementarily when M is the 3-sphere S^3 and

$$g=\begin{bmatrix}\cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & \multicolumn{2}{c}{0} \\ & & \cos\varphi & -\sin\varphi \\ \multicolumn{2}{c}{0} & \sin\varphi & \cos\varphi\end{bmatrix}\in SO(4)$$

by determining the basis for the eigenspace of A. Our result is the following equation:

$$\eta_A(g,s)=-\frac{2\sin\varphi}{\cos\theta-\cos\varphi}\sum_{k=0}^{\infty}\frac{\sin(k+2)\theta}{(k+2)^s}-\frac{2\sin\theta}{\cos\varphi-\cos\theta}\sum_{k=0}^{\infty}\frac{\sin(k+2)\varphi}{(k+2)^s}.$$

Details and further arguments will be given elsewhere.

1. Let $H_k(\boldsymbol{R}^{n+1})$ be the vector space over $\boldsymbol{R}$ consisting of all harmonic homogeneous polynomials of degree k on $\boldsymbol{R}^{n+1}$ and let $H_k^p(\boldsymbol{R}^{n+1})$ be the vector space over $\boldsymbol{R}$ consisting of all p-forms on $\boldsymbol{R}^{n+1}$ of which coefficients are the elements in $H_k(\boldsymbol{R}^{n+1})$. It is known that the inclusion map $\iota: S^n \to \boldsymbol{R}^{n+1}$ induces the isomorphism

$$\iota^*: H_k^p(\boldsymbol{R}^{n+1}) \cap \operatorname{Ker} d_0 \cap \operatorname{Ker} \delta_0 \longrightarrow V_{\lambda_k}^p(S^n) \cap \operatorname{Ker} d$$

where d_0 and δ_0 are the exterior differential and its codifferential on the space $\wedge^*(\boldsymbol{R}^{n+1})$ of differential forms on $\boldsymbol{R}^{n+1}$ respectively, and $V_{\lambda_k}^p(S^n)$ is the eigenspace of the Laplace operator Δ on $\wedge^p(S^n)$ with eigenvalue $\lambda_k = (k+p)(n+k-p+1)$. (See A. Ikeda and Y. Taniguchi [3].)

Since $A^2 = \Delta$, we know that the eigenvalues of A are of the form $\pm\sqrt{\lambda_k}$ and there exists an isomorphism

$$f: \operatorname{Ker}(\Delta - \lambda_k) \longrightarrow \operatorname{Ker}(A - \sqrt{\lambda_k}) \oplus \operatorname{Ker}(A + \sqrt{\lambda_k})$$

given by $f(v) = ((A+\sqrt{\lambda_k})v, (A-\sqrt{\lambda_k})v)$. Note that the multiplicities of $\sqrt{\lambda_k}$ and $-\sqrt{\lambda_k}$ are equal, i.e., $\operatorname{Ker}(A-\sqrt{\lambda_k})$ is isomorphic to $\operatorname{Ker}(A+\sqrt{\lambda_k})$.

Moreover, there exist a direct sum decomposition

$$V_{\lambda_k}^p(S^n) = (V_{\lambda_k}^p(S^n) \cap \operatorname{Ker} d) \oplus (V_{\lambda_k}^p(S^n) \cap \operatorname{Ker} \delta)$$

and an isomorphism

$$d: V_{\lambda_k}^p(S^n) \cap \operatorname{Ker} \delta \longrightarrow V_{\lambda_k}^{p+1}(S^n) \cap \operatorname{Ker} d$$

(see [3]).

Thus, to calculate the eta-function on S^3, we have only to consider

$$A|_{V_{\lambda_k}^2(S^3) \cap \operatorname{Ker} d} = d*: V_{\lambda_k}^2(S^3) \cap \operatorname{Ker} d \longrightarrow V_{\lambda_k}^2(S^3) \cap \operatorname{Ker} d,$$

where $\lambda_k = (k+2)^2$, and investigate the trace of g acting on $V_{\lambda_k}^2(S^3) \cap \operatorname{Ker} d$.

2. Let $\Delta_0 = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_{n+1})^2$ be the Laplace operator on $\boldsymbol{R}^{n+1}$.

Let a homogeneous polynomial $p(x_1, \cdots, x_{n+1})$ of degree k be given by

$$p(x_1, \cdots, x_{n+1}) = p_k(x_2, \cdots, x_{n+1}) + x_1 p_{k-1}(x_2, \cdots, x_{n+1}) + \cdots + x_1^{k-1} p_1(x_2, \cdots, x_{n+1}) + x_1^k p_0,$$

where $p_{k-i}(x_2, \cdots, x_{n+1})$ is a homogeneous polynomial of degree $k-i$ on $x_2, \cdots, x_{n+1}$. Then,

$$\Delta_0 p(x_1, \cdots, x_{n+1}) = \sum_{i=0}^{k-2} \{(i+2)(i+1) p_{k-i-2}(x_2, \cdots, x_{n+1}) + \Delta_0' p_{k-i}(x_2, \cdots, x_{n+1})\} x_1^i,$$

where $\Delta_0' = (\partial/\partial x_2)^2 + \cdots + (\partial/\partial x_{n+1})^2$.

Hence, $p(x_1, \cdots, x_{n+1}) \in H_k(\boldsymbol{R}^{n+1})$ if and only if

$$(i+2)(i+1) p_{k-i-2}(x_2, \cdots, x_{n+1}) + \Delta_0' p_{k-i}(x_2, \cdots, x_{n+1}) = 0$$

for $i = 0, 1, \cdots, k-2$. That is to say,

$$p_{k-2i}(x_2, \cdots, x_{n+1}) = \frac{(-1)^i}{(2i)!} (\Delta_0')^i p_k(x_2, \cdots, x_{n+1})$$

and

$$p_{k-2i-1}(x_2, \cdots, x_{n+1}) = \frac{(-1)^i}{(2i+1)!} (\Delta_0')^i p_{k-1}(x_2, \cdots, x_{n+1})$$

for $i = 1, 2, \cdots, [k/2]$ and hence any element in $H_k(\boldsymbol{R}^{n+1})$ is determined by $p_k(x_2, \cdots, x_{n+1})$ and $p_{k-1}(x_2, \cdots, x_{n+1})$.

Therefore we have a canonical basis

$$\{h^k_{i_2,\cdots,i_{n+1}}, \bar{h}^k_{j_2,\cdots,j_{n+1}} | i_2+\cdots+i_{n+1}=k,\ j_2+\cdots+j_{n+1}=k-1\}$$

for $H_k(\boldsymbol{R}^{n+1})$ where

$$h^k_{i_2,\cdots,i_{n+1}} = x_2^{i_2}\cdots x_{n+1}^{i_{n+1}} - \frac{1}{2!}x_1^2 \Delta_0'(x_2^{i_2}\cdots x_{n+1}^{i_{n+1}}) + \frac{1}{4!}x_1^4(\Delta_0')^2(x_2^{i_2}\cdots x_{n+1}^{i_{n+1}}) - \cdots$$

and

$$\bar{h}^k_{j_2,\cdots,j_{n+1}} = x_1 x_2^{j_2}\cdots x_{n+1}^{j_{n+1}} - \frac{1}{3!}x_1^3 \Delta_0'(x_2^{j_2}\cdots x_{n+1}^{j_{n+1}}) + \frac{1}{5!}x_1^5(\Delta_0')^2(x_2^{j_2}\cdots x_{n+1}^{j_{n+1}}) - \cdots.$$

Note that

$$\dim H_k(\boldsymbol{R}^{n+1}) = {}_n\mathrm{H}_k + {}_n\mathrm{H}_{k-1} = \binom{n+k-1}{k} + \binom{n+k-2}{k-1} = \frac{(n+2k-1)\cdot(n+k-2)!}{k!\,(n-1)!}.$$

As for the derivatives, we have

Lemma 1.

$$\frac{\partial}{\partial x_1} h^k_{i_2,\cdots,i_{n+1}} = -\sum_{s=2}^{n+1} i_s(i_s-1)\bar{h}^{k-1}_{i_2,\cdots,i_s-2,\cdots,i_{n+1}}$$

$$\frac{\partial}{\partial x_1}\bar{h}^k_{j_2,\cdots,j_{n+1}} = h^{k-1}_{j_2,\cdots,j_{n+1}}$$

$$\frac{\partial}{\partial x_u} h^k_{i_2,\cdots,i_{n+1}} = i_u h^{k-1}_{i_2,\cdots,i_u-1,\cdots,i_{n+1}} \qquad u=2,3,\cdots,n+1$$

$$\frac{\partial}{\partial x_u}\bar{h}^k_{j_2,\cdots,j_{n+1}} = j_u \bar{h}^{k-1}_{j_2,\cdots,j_u-1,\cdots,j_{n+1}} \qquad u=2,3,\cdots,n+1.$$

3. Let

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & \multicolumn{2}{c}{0} \\ & & \cos\varphi & -\sin\varphi \\ \multicolumn{2}{c}{0} & \sin\varphi & \cos\varphi \end{bmatrix} \in SO(4)$$

act on $\boldsymbol{R}^4 = \boldsymbol{R}^2 \oplus \boldsymbol{R}^2$. Then g induces the action on $\wedge^*(\boldsymbol{R}^4)$ and $\wedge^*(S^3)$ also denoted by g. In particular, we have

Lemma 2.

$$g\cdot(\pm dx_1dx_2 + dx_3dx_4) = \pm dx_1dx_2 + dx_3dx_4$$

$$g\cdot(\mp dx_1dx_3 + dx_2dx_4) = \cos(\varphi\pm\theta)(\mp dx_1dx_3 + dx_2dx_4) + \sin(\varphi\pm\theta)(\pm dx_1dx_4 + dx_2dx_3)$$

$$g\cdot(\pm dx_1dx_4 + dx_2dx_3) = -\sin(\varphi\mp\theta)(\mp dx_1dx_3 + dx_2dx_4) + \cos(\varphi\pm\theta)(\pm dx_1dx_4 + dx_2dx_3)$$

Considering Lemmas 1 and 2, we can choose a suitable basis for $H^2_k(\boldsymbol{R}^4) \cap \operatorname{Ker} d_0 \cap \operatorname{Ker} \delta_0$ to calculate the trace of g.

Let

$$\alpha^\pm_{k-i,i,0} = \pm i\bar{h}^k_{k-i,i-1,0}(\pm dx_1dx_2 + dx_3dx_4) \pm (k-i)\bar{h}^k_{k-i-1,i,0}(\mp dx_1dx_3 + dx_2dx_4) + h^k_{k-i,i,0}(\pm dx_1dx_4 + dx_2dx_3) \qquad i=0,1,\cdots,k$$

$$\beta^\pm_{k-i,0,i} = \pm i\bar{h}^k_{k-i,0,i-1}(\pm dx_1dx_2 + dx_3dx_4) + h^k_{k-i,0,i}(\mp dx_1dx_3 + dx_2dx_4) \mp (k-i)\bar{h}^k_{k-i-1,0,i}(\pm dx_1dx_4 + dx_2dx_3) \qquad i=0,1,\cdots,k$$

$$\gamma^\pm_{0,k-i,i} = h^k_{0,k-i,i}(\pm dx_1dx_2 + dx_3dx_4) \mp i\bar{h}^k_{0,k-i,i-1}(\mp dx_1dx_3 + dx_2dx_4)$$

$$\mp(k-i)\bar{h}^k_{0,k-i-1,i}(\pm dx_1dx_4+dx_2dx_3) \qquad i=0,1,\cdots,k$$

$$\begin{aligned}\delta^{\pm}_{k-i-j-1,i,j}=&\{(i+1)h^k_{k-i-j,i,j}\pm(k-i-j)j\bar{h}^k_{k-i-j-1,i+1,j-1}\}(\pm dx_1dx_2+dx_3dx_4)\\&+\{(k-i-j)h^k_{k-i-j-1,i+1,j}\mp(i+1)j\bar{h}^k_{k-i-j,i,j-1}\}\\&\qquad\times(\mp dx_1dx_3+dx_2dx_4)\\&\mp\{(i+1)i\bar{h}^k_{k-i-j,i-1,j}+(k-i-j)(k-i-j-1)\bar{h}^k_{k-i-j-2,i+1,j}\}\\&\qquad\times(\pm dx_1dx_4+dx_2dx_3) \qquad 0\leqq i,j \text{ and } i+j\leqq k-1\end{aligned}$$

$$\begin{aligned}\varepsilon^{\pm}_{k-i-j-1,i,j}=&\pm\{(i+1)i\bar{h}^k_{k-i-j-1,i-1,j+1}+(j+1)j\bar{h}^k_{k-i-j-1,i+1,j-1}\}\\&\qquad\times(\pm dx_1dx_2+dx_3dx_4)\\&+\{(j+1)h^k_{k-i-j-1,i+1,j}\pm(i+1)(k-i-j-1)\bar{h}^k_{k-i-j-2,i,j+1}\}\\&\qquad\times(\mp dx_1dx_3+dx_2dx_4)\\&+\{(i+1)h^k_{k-i-j-1,i,j+1}\mp(j+1)(k-i-j-1)\bar{h}^k_{k-i-j-2,i+1,j}\}\\&\qquad\times(\pm dx_1dx_4+dx_2dx_3) \qquad 0\leqq i,j \text{ and } i+j\leqq k-1.\end{aligned}$$

Then, these are linearly independent and the number of $\alpha^{\pm},\cdots,\varepsilon^{\pm}$'s is $(k+1)\times3\times2+k(k+1)/2\times2\times2=2(k+1)(k+3)$ which coincides with the dimension of $H^2_k(\mathbf{R}^4)\cap\operatorname{Ker}d_0\cap\operatorname{Ker}\delta_0$ (see I. Iwasaki and K. Katase [4]). Thus we have the following

Lemma 3. *$\{\alpha^{\pm},\beta^{\pm},\cdots,\varepsilon^{\pm}\}$ is a basis for $H^2_k(\mathbf{R}^4)\cap\operatorname{Ker}d_0\cap\operatorname{Ker}\delta_0$ and hence $\{\iota^*\alpha^{\pm},\iota^*\beta^{\pm},\cdots,\iota^*\varepsilon^{\pm}\}$ is a basis for $V^2_{\lambda_k}(S^3)\cap\operatorname{Ker}d$.*

Remark. The double signature $\pm$ corresponds to the element of $\operatorname{Ker}(A\mp\sqrt{\lambda_k})$ respectively.

Now the diagonal components of the action of g on $V^2_{\lambda_k}(S^3)\cap\operatorname{Ker}d$ are as follows:

$\iota^*\alpha^{\pm}_{k-i,i,0}$	$\cos^i\varphi\cos(\varphi\pm(k-i+1)\theta)$
$\iota^*\beta^{\pm}_{k-i,0,i}$	$\cos^i\varphi\cos(\varphi\pm(k-i+1)\theta)$
$\iota^*\gamma^{\pm}_{0,k-i,i}$	$\cos^{k-i}\varphi\sin^i\varphi$
$\iota^*\delta^{\pm}_{k-i-j-1,i,j}$	$\cos(k-i-j)\theta\sum_{s=0}^{\text{Min.}(i,j)}(-1)^s\binom{i}{s}\binom{j}{s}\sin^{2s}\varphi\cos^{i+j-2s}\varphi$ $\pm\sin(k-i-j)\theta\sum_{s=0}^{\text{Min.}(i,j-1)}(-1)^s\binom{i}{s}\binom{j}{s+1}\sin^{2s+1}\varphi\cos^{i+j-2s-1}\varphi$
$\iota^*\varepsilon^{\pm}_{k-i-j-1,i,j}$	$\cos(\varphi\pm(k-i-j)\theta)\sum_{s=0}^{\text{Min.}(i,j+1)}(-1)^s\binom{i}{s}\binom{j+1}{s}\sin^{2s}\varphi\cos^{i+j-2s+1}\varphi$ $-\sin(\varphi\pm(k-i-j)\theta)\sum_{s=0}^{\text{Min.}(i,j)}(-1)^s\binom{i}{s}\binom{j+1}{s+1}\sin^{2s+1}\varphi\cos^{i+j-2s}\varphi$

Hence we have

$$\begin{aligned}Tr(g\,|\,E_{\sqrt{\lambda_k}})-Tr(g\,|\,E_{-\sqrt{\lambda_k}})=&2\Big\{-2\sum_{i=0}^{k}\cos^i\varphi\sin\varphi\sin(k-i+1)\varphi\Big\}\\&+2\sum_{i+j\leqq k-1}\sin(k-i-j)\theta\sum_{s=0}^{\text{Min.}(i,j-1)}(-1)^s\binom{i}{s}\binom{j}{s+1}\sin^{2s+1}\varphi\cos^{i+j-2s-1}\varphi\\&-2\sum_{i+j\leqq k-1}\sin\varphi\sin(k-i-j)\theta\sum_{s=0}^{\text{Min.}(i,j+1)}(-1)^s\binom{i}{s}\binom{j+1}{s}\sin^{2s}\varphi\cos^{i+j-1-2s}\varphi\\&-2\sum_{i+j\leqq k-1}\cos\varphi\sin(k-i-j)\theta\sum_{s=0}^{\text{Min.}(i,j)}(-1)^s\binom{i}{s}\binom{j+1}{s+1}\sin^{2s+1}\varphi\cos^{i+j-2s}\varphi\\&=-4\{\sin\varphi\sin(k+1)\theta+\sin2\varphi\sin k\theta+\cdots+\sin(k+1)\varphi\sin\theta\}\end{aligned}$$

and we obtain the following

Proposition.

$$\eta_A(g,s)=-4\sum_{k=0}^{\infty}\frac{\sum_{i=1}^{k+1}\sin i\varphi\sin(k+2-i)\theta}{(k+2)^s}$$

$$=-\frac{2\sin\varphi}{\cos\theta-\cos\varphi}\sum_{k=0}^{\infty}\frac{\sin(k+2)\theta}{(k+2)^s}-\frac{2\sin\theta}{\cos\varphi-\cos\theta}\sum_{k=0}^{\infty}\frac{\sin(k+2)\varphi}{(k+2)^s}.$$

Note that the infinite series $\sum_{k=0}^{\infty}\sin(k+2)x/(k+2)^s$ is uniformly convergent on the closed interval except the points $x=n\pi$, $n=0, \pm 1, \pm 2, \cdots$ (see, for example, [2]).

Remarks. 1. Letting $s\to 0$, we have

$$\eta_A(g,0)=-\frac{2\sin\varphi}{\cos\theta-\cos\varphi}\left(\frac{1}{2}\cot\frac{\theta}{2}-\sin\theta\right)$$

$$-\frac{2\sin\theta}{\cos\varphi-\cos\theta}\left(\frac{1}{2}\cot\frac{\varphi}{2}-\sin\varphi\right)$$

$$=-\cot\frac{\theta}{2}\cot\frac{\varphi}{2}$$

which coincides with the result of Atiyah, Patodi and Singer [1].

2. With a little more computation, we can obtain the eta-function on the odd dimensional sphere (see I. Iwasaki [5]).

References

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