37. *A Further Generalization of the Ostrowski Theorem in Banach Spaces*

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§1. Let $f: D\subset R^n\to R^n$ be Fréchet differentiable at an interior point x^* of D and $f(x^*)=x^*$. If the spectral radius of $f'(x^*)$ satisfies $\rho(f'(x^*))<1$, then x^* is a point of attraction (or an attractor) of the iterates $f(x_k)=x_{k+1}$, i.e., there is an open neighborhood S of x^* such that $S\subset D$ and, for any $x_0\in S$, the iterates $\{x_k\}$ defined by $f(x_k)=x_{k+1}$ all lie in D and converge to x^*. The sufficiency of $\rho(f'(x^*))<1$ for a point of attraction was proved by Ostrowski [4, pp. 118–120] (first edition) under somewhat more stringent condition on f, and later by Ostrowski [4, pp. 161–164] (second edition) and [5, pp. 150–152] under those of the above theorem. Using the well known spectral radius formula in Banach algebra, Kitchen [3] extended Ostrowski's theorem to an arbitrary Banach space. Ostrowski's theorem occupies a special place in the study of Newton's iteration processes [4]. To study non-stationary (nonautonomous) processes and Newton-SOR processes, Ortega and Rheinboldt [4, pp. 349–350] extended Ostrowski's theorem in a more general form. Generalizing further, we shall extend this general form to an arbitrary Banach space.

§2. Let X and Y be two real Banach spaces. A family of maps $\{f_h\}$, where $f_h: D\subset X\to X$ and the parameter vector h varies over some set $D_h\subset Y$, is uniformly Fréchet differentiable at an interior point of D if each f_h is Fréchet differentiable at an interior point of D if each f_h is Fréchet differentiable at x and if for any $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$, independent of h, such that $S(x,\delta)=\{y\in X: \|y-x\|<\delta\}\subset D$ and

$$\|f_h(y)-f_h(x)-f'_h(x)(y-x)\|\leqq\varepsilon\|y-x\|$$

for all $y\in S(x,\delta)$ and for all $h\in D_h$.

Theorem (Generalized Ostrowski theorem in Banach spaces). *Let X and Y be two real Banach spaces. For $f: D\times D_h\subset X\times Y\to X$ and x^* is an interior point of D such that $x^*=f(x^*,h)$ for all $h\in D_h$, assume that the family of maps $\{f_h\}$, where*

$$f_h: D\subset X\to X,\ f_h(x)=f(x,h),\ x\in D,\ h\in D_h,$$

is uniformly Fréchet differentiable at x^* for all $h\in D_h$, and that

$$f'_h(x^*)=H^{q(h)},\qquad \text{for all } h\in D_h,$$

where H is a bounded linear operator on X satisfies $\rho(H)<1$ and $q(h)$

is a positive integer. Then there is an open neighborhood S of x^* such that for any $x_0 \in S$ and any sequence $\{h_k\} \subset D_h$ the iterates $\{x_k\}$ given by

$$x_{k+1} = f(x_k, h_k), \qquad k = 0, 1, \cdots,$$

are well defined and converge to x^*.

To prove the theorem, we shall apply the following remarkable infinite dimensional result which is due to Holmes [1]. By virtue of this result, our proof of the theorem is different from Kitchen's method. It should be noted that finite dimensional case of the result was given by Householder [2, p. 46] (see Ortega and Rheinboldt [4, p. 44] for a transparent proof).

Lemma. *Let T be a bounded linear operator on a Banach space X. Then, given any $\varepsilon > 0$, there is a norm $\|\cdot\|$ equivalent to the given norm on X such that*

$$\|T\| \leqq \rho(T) + \varepsilon.$$

§ 3. Proof of Theorem. Set $\sigma = \rho(H)$ and take $\varepsilon > 0$. The above lemma ensures of a norm on X for which

$$\|H\| \leqq \sigma + \varepsilon.$$

In this norm, the uniform Fréchet differentiability of the family of maps $\{f_h\}$ allows one to choose $\delta = \delta(\varepsilon) > 0$ so that $S = S(x^*, \delta) \subset D$ and

$$\begin{aligned} \|f(x, h) - x^*\| &\leqq \|f_h(x) - f_h(x^*) - f'_h(x^*)(x - x^*)\| + \|f'_h(x^*)(x - x^*)\| \\ &\leqq \varepsilon \|x - x^*\| + \|H^{q(h)}\| \|x - x^*\| \\ &\leqq [\varepsilon + (\sigma + \varepsilon)^{q(h)}] \|x - x^*\| \end{aligned}$$

whenever $x \in S$ and $h \in D_h$. Since $\sigma < 1$, we may assume that $\varepsilon > 0$ is chosen so that $\sigma + 2\varepsilon < 1$. Then $q(h) \geqq 1$ implies that

$$\varepsilon + (\sigma + \varepsilon)^{q(h)} \leqq \sigma + 2\varepsilon \equiv \alpha < 1.$$

Hence, if $x_0 \in S$, then

$$\|x_1 - x^*\| = \|f(x_0, h_0) - x^*\| \leqq \alpha \|x_0 - x^*\|.$$

Therefore, $x_1 \in S$, and it follows by induction that all x_k are in S and, moreover, that

$$\|x_k - x^*\| \leqq \alpha \|x_{k-1} - x^*\| \leqq \cdots \leqq \alpha^k \|x_0 - x^*\|.$$

Thus, $x_k \to x^*$ as $k \to \infty$, and the proof is complete.

When $D_h = \{h\}$ is a singleton and $q(h) \equiv 1$, the theorem reduces to Kitchen's result and that for $D_h = \{0, 1, 2, \cdots\}$ in R^1 and $h_k \equiv k$, the iteration $x_{k+1} = f(x_k, h_k)$ is simply one-step nonstationary (nonautonomous) process

$$x_{k+1} = f_k(x_k), \quad k = 0, 1, \cdots, \quad \text{where } f_k(\cdot) = f(\cdot, k).$$

Note that $\rho(H) < 1$ cannot be replaced by letting $\rho(f'_h(x^*)) < 1$ for all $h \in D_h$, as the following two-dimensional example shows:

$$\begin{aligned} f_k(x) &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x, \qquad \text{if } k \text{ is even,} \\ &= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} x, \qquad \text{if } k \text{ is odd.} \end{aligned}$$

Then $\rho(f_k'(0))=0$, $k=0, 1, \cdots$, but 0 is not a point of attraction of the iterates $f_k(x_k)=x_{k+1}$, $k=0, 1, \cdots$.

References

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