29. On the Regularity of Arithmetic Multiplicative Functions. II

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In our previous paper ([1]) we discussed some sufficient conditions under which an arithmetic multiplicative function turns out to be completely multiplicative. In this paper we shall extend and refine the previous results and make some remarks about remaining problems in this field.

1. Let S be a sequence in N of density zero, and $\mathcal{C}_S$ be the set of all those sequences $\{a_n\}_{n=1}^{\infty}$ which satisfy the condition

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\leqslant x\\ n\notin S}}|a_n|=0.$$

If $S=\phi$, we abbreviate $\mathcal{C}_S$ to $\mathcal{C}$.

Theorem. *Let $F(n)$ and $G(n)$ be arithmetic multiplicative functions, a and b be positive integers and $(a, b)=1$, ε be either $+1$ or -1, fixed arbitrary. Suppose $|F(an+\varepsilon b)|=1$ if n and $an+\varepsilon b \in N$, $|G(n)|=1$ for any $n\in N$, and $\{F(an+\varepsilon b)-C\cdot G(n)\}_{n=1}^{\infty}\in \mathcal{C}_S$ for some S and for some constant C.*

I) *When a is even, we can decompose $F(n)$ and $G(n)$:*

$G(n)=G'(n)\cdot H(n)$ *for any* $n\in N$,

$F(n)=G'(n)\cdot H(n)$ *for any n such that* $(n, a)=1$,

where $G'(n)$ and $H(n)$ are multiplicative functions satisfying

i) *$G'(n)$ is completely multiplicative,*

ii) $H(n)=H((n, b))$ *for any* $n\in N$,

iii) $\{G'(kn+\varepsilon)-G'(k)\cdot G'(n)\}_{n=1}^{\infty}\in \mathcal{C}$, *for any* $k\geqslant a$.

Further we have

$$C=G'(a).$$

II) *When a is odd, suppose $2^{\alpha}\| b$ $(\alpha\geqslant 0)$. We can decompose $F(n)$ and $G(n)$:*

$G(n)=G'(n)\cdot H(n)\cdot H'_{\alpha}(n)$ *for any* $n\in N$,

$F(n)=G'(n)\cdot H(n)\cdot \overline{H'_{\alpha}(n)}$ *for any n such that* $(n, a)=1$,

where $\bar{z}$ denotes the complex conjugate of z, $G'(n)$ and $H(n)$ are multiplicative functions satisfying the above i)–iii), *and $H'_{\alpha}(n)$ is the multiplicative function which is defined as follows;*

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$$H_0'(n)=\begin{cases} 1 & \text{if } n \text{ is odd,} \\ z_0 & \text{if } n \text{ is even,} \end{cases}$$

and for $\alpha \geqslant 1$,

$$H_\alpha'(n)=\begin{cases} 1 & \text{if } 2^\alpha \nmid n, \\ z_\alpha & \text{if } 2^\alpha \| n, \\ \bar{z}_\alpha & \text{if } 2^{\alpha+1} | n, \end{cases}$$

where any of z_α ($\alpha \geqslant 0$) *is a computable constant of modulus* 1 *and depending only on* α *and some special values of* $G(n)$. *Further we have*

$$C=\begin{cases} G'(a) & \text{if } b \text{ is even,} \\ G'(a)\cdot G(2^{\alpha+2})\cdot G(2^\alpha)\cdot \overline{G^2(2^{\alpha+1})} & \text{if } b \text{ is odd.} \end{cases}$$

From this theorem we can deduce, by a similar method as in our Part I, a theorem concerning additive functions:

Corollary. *Let* $f(n)$ *and* $g(n)$ *be additive arithmetic functions,* a *and* b *be positive integers and* $(a, b)=1$, *and* ε *be either* $+1$ *or* -1, *fixed arbitrary, and* c *be a constant. Suppose* $\{f(an+\varepsilon b)-g(n)-c\}_{n=1}^{\infty} \in \mathcal{C}_S$.

I) *When* a *is even, we can decompose* $f(n)$ *and* $g(n)$:

$$g(n)=g'(n)+h(n) \qquad \text{for any } n \in \boldsymbol{N},$$
$$f(n)=g'(n)+h(n) \qquad \text{for any } n \text{ such that } (n, a)=1.$$

where $g'(n)$ *and* $h(n)$ *are additive functions satisfying*

i′) $g'(n)$ *is completely additive,*

ii′) $h(n)=h((n, b))$ *for any* $n \in \boldsymbol{N}$.

Further we have

$$c=g'(a).$$

II) *When* a *is odd, suppose* $2^\alpha \| b$ ($\alpha \geqslant 0$). *We can decompose* $f(n)$ *and* $g(n)$:

$$g(n)=g'(n)+h(n)+h_\alpha'(n) \qquad \text{for any } n \in \boldsymbol{N},$$
$$f(n)=g'(n)+h(n)-h_\alpha'(n) \qquad \text{for any } n \text{ such that } (n, a)=1,$$

where $g'(n)$ *and* $h(n)$ *are additive functions satisfying the above* i′) *and* ii′), *and* $h'(n)$ *is also the additive function which is defined as follows;*

$$h_0'(n)=\begin{cases} 0 & \text{if } n \text{ is odd,} \\ y_0 & \text{if } n \text{ is even,} \end{cases}$$

and for $\alpha \geqslant 1$,

$$h_\alpha'(n)=\begin{cases} 0 & \text{if } 2^\alpha \nmid n, \\ y_\alpha & \text{if } 2^\alpha \| n, \\ -y_\alpha & \text{if } 2^{\alpha+1} | n, \end{cases}$$

where $y_\alpha (\alpha \geq 0)$ *is a computable constant depending only on* α *and some special values of* $g(n)$. *Further we have*

$$c=\begin{cases} g'(a) & \text{if } b \text{ is even,} \\ g'(a)+g(2^{\alpha+2})+g(2^\alpha)-2g(2^{\alpha+1}) & \text{if } b \text{ is odd.} \end{cases}$$

2. We shall show here only some crucial points for proving these results.

Lemma 1. *Under the assumptions of our theorem, we have* $\{F(an+\varepsilon b)-C\cdot G(n)\}_{n=1}^{\infty}\in \mathcal{C}$.

Lemma 2. *Let* $F(n)$ *be a multiplicative function such that* $\{F(d^2n+\varepsilon)-C\cdot F(dn)\}_{n=1}^{\infty}\in \mathcal{C}$, *then* $F'(n)=F(dn)\cdot\overline{F(d)}$ *is completely multiplicative and* $C=F'^2(d)\cdot\overline{F(d)}$.

Lemma 3. *Let* $H(n)$ *be completely multiplicative function such that* $\{H(\beta n+\varepsilon)-C\cdot H(n)\}_{n=1}^{\infty}\in \mathcal{C}$, *then we get*

$$C=H(\beta),$$

$$\{H(\gamma n+\varepsilon)-H(\gamma n)\}_{n=1}^{\infty}\in \mathcal{C} \qquad \textit{for any } \gamma\geqslant\beta,$$

and

$$\{H(n(vv_0+1)+\varepsilon v_0)-H(n(vv_0+1))\}_{n=1}^{\infty}\in \mathcal{C} \qquad \textit{for any } v\geqslant\beta \textit{ and any } v_0\geqslant 1.$$

The proof of Lemma 1 is trivial, but by this lemma we can neglect an influence of the sequence S; this is why we utilize multiplicative functions. The proofs of Lemmas 2 and 3 are essential parts of the proof of our Theorem. These proofs are carried out only by help of elementary facts, but in total they are complicated; the main part of them is shown in [2].

3. Heretofore, in our Parts I and II, we have restricted ourselves to the problem; under what sort of conditions multiplicative functions turn out to be *completely* multiplicative, while we have another type of problem; under what sort of conditions we can determine the form of multiplicative (resp. additive) functions.

Concerning this problem, I. Kátai ([3]) obtained the following result; if an additive function $f(n)$ satisfies $\{f(n+1)-f(n)\}_{n=1}^{\infty}\in \mathcal{C}$, then $f(n)$ is a constant multiple of $\log n$. Besides this, we do not know much about this problem.

In this connection, we propose here the following three problems.

Problem 1. If an additive function $f(n)$ satisfies

$$\{f(n+1)-f(n)\}_{n=1}^{\infty}\in \mathcal{C}_S,$$

where S is not finite, can we deduce $f(n)=c\log n$?

Problem 2. If an additive function $f(n)$ satisfies

$$\{f(an+\varepsilon)-f(n)-f(a)\}_{n=1}^{\infty}\in \mathcal{C}$$

for some positive integer $a\geqslant 2$, can we deduce $f(n)=c\log n$?

Problem 3. If a multiplicative function $F(n)$ satisfies

$$|F(n)|=1, \quad \{F(n+1)-F(n)\}_{n=1}^{\infty}\in \mathcal{C},$$

and has a mean-value equal to 0, what can we deduce about the form of $F(n)$?

In addition, we notice that the first of us proved that, if an additive function $f(n)$ satisfies both

$$\{f(an+1)-f(n)-f(a)\}_{n=1}^{\infty}\in \mathcal{C}$$

and

$$\{f(an-1)-f(n)-f(a)\}_{n=1}^{\infty}\in \mathcal{C},$$

then $f(n)=c\log n$.

References

[1] J.-L. Mauclaire and L. Murata: On the regularity of arithmetic multiplicative functions. I. Proc. Japan Acad., **56A**, 438–440 (1980).
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[3] I. Kátai: On a problem of P. Erdös. J. of Number Theory, **2**, 1–6 (1970).