14. *On a Representation of the Solution of the Cauchy Problem with Singular Initial Data*

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1. Introduction. In this note we shall deal with the Cauchy problem with singular initial data in the complex domain for partial differential operators with holomorphic characteristic roots. This problem was introduced by Hamada [1] and developed by Hamada-Leray-Wagschal [2], Nakamura [4] and Hamada-Nakamura [5]. Recently, Kumano-go-Taniguchi [3] constructed the fundamental solution for a hyperbolic system in the real variables' case. Their form of the solution is suggestive to ours. In fact their multiphase function corresponds to our multicharacteristic function, and so does their infinite sum of iterated integral. However, our construction of the solution is quite different from theirs; we solve the transport equations.

2. Notations and result. Let $H(t, x; \partial_t, \partial_x)$ be a partial differential operator of order m whose coefficients are holomorphic in

$$\Omega=\{|t|+\textstyle\sum_{i=1}^n |x_i|<R\}\subset \mathbf{C}^{n+1} \qquad (R>0),$$

and the coefficient of ∂_t^m is identically 1. We impose the following condition on the principal part $h(t, x; \partial_t, \partial_x)$ of $H(t, x; \partial_t, \partial_x)$:

(A) $h(t, x; \tau, \zeta)$ is written as $h(t, x; \tau, \zeta)=\prod_{j=1}^m(\tau+\lambda_j(t, x; \zeta))$, where each $\lambda_j(t, x; \zeta)$ is holomorphic in $\Omega\times\Omega^*=\Omega\times\{\sum_{i=1}^n|\zeta_i-\delta_{1,i}|<R_1\}$ $(R_1>0)$.

We shall consider the following Cauchy problem with singular data in Ω:

$$\text{(E)} \qquad H(t, x; \partial_t, \partial_x)u(t, x)=0;$$

$$\text{(C)} \qquad (\partial_t)^l u(0, x)=\frac{c_l(x')}{x_1^{p_l}} \qquad (l=0, 1, \cdots, m-1),$$

where $c_l(x')$ is holomorphic in $\{\sum_{i=2}^n |x_i|<R\}$, and p_l a positive integer.

We shall construct the solution $u(t, x)$ of (E), (C) in the form

$$\text{(U)} \qquad \begin{aligned} u(t, x)=&\textstyle\sum_{j=j_1}^\infty f_j(\phi_1(t, x))a_{j,1}(t, x)\\ &+\textstyle\sum_{k=2}^\infty \int_0^t dt_1\int_0^{t-t_1}dt_2\cdots\int_0^{t-t_1-\cdots-t_{k-2}}dt_{k-1}\\ &\times\textstyle\sum_{j=j_k}^\infty f_j(\phi_k((t)_k, x))a_{j,k}((t)_k, x).\end{aligned}$$

Here $(t)_k=(t_1, \cdots, t_k)\in C^k$, $t_1+\cdots+t_k=t$; for $s\in C$, $f_j(s)=(s^j/j!)(\log s-\Gamma'(j+1)/\Gamma(j+1))$ for $j\geq 0$ and $f_j(s)=(-1)^{j+1}(-j-1)!\,s^j$ for $j\leq -1$. $\phi_k((t)_k, x)$ and $a_{j,k}((t)_k, x)$ $(k\in N, j\in Z)$ are holomorphic at the origin of $C^k\times C^n$. The $(k-1)$-*ple* integral is the iterated integral along the segment from 0 to t in the complex plane, whose variables are $0, t_1, t_1+t_2, \cdots, t_1+\cdots+t_k=t$ in order (see the following):

$$0\cdot\underset{t_1}{—\cdot}\underset{t_1+t_2\cdots}{———\cdot}\underset{t_1+t_2+\cdots+t_{k-1}}{————\cdot}——\cdot t.$$

Our result can be stated as follows:

Theorem. *We can find a family of functions ϕ_k, $a_{j,k}$ in* (U), *and a neighbourhood $\tilde{\Omega}$ of the origin of C^{n+1}, satisfying the following statements*:

(1) (i) $\phi_k((t)_k, x) \ll \Phi(K(t_1+\cdots+t_k)+x_1+\cdots+x_n)$;

(ii) $a_{j,k}((t)_k, x)=0$ for $N(j_1, k_1)<0$, and

$$a_{j,k}((t)_k, x) \ll C_0 C_1^{j_1+k_1} \eta_{N(j_1,k_1)}(R_3, r, \rho(t_1+\cdots+t_k)+x_1+\cdots+x_n) \quad \text{for } N(j_1, k_1)\geq 0,$$

where Φ is a holomorphic function in $|s|<R_2$ and K, R_3, r, ρ are positive numbers, all of which are independent of k. $j_1=j+j_0$, $j_0=\max_{0\leq l\leq m-1}(m+p_l-l)$, $k_1=k-m-1$; $N(j_1,k_1)=k_1+j_1-[(m-2-j_1)/(m-1)]$ for $j_1\leq 0$ and $N(j_1,k_1)=k_1+j_1$ for $j_1>0$, and $\eta_N(R,r,s)=(d/ds)^N(R/R-s)(1/r-s)$.

(2) *For any $(0,x)\in\tilde{\Omega}$, $x_1\neq 0$, there exists a $\delta=\delta(x)>0$ such that for any $|t|<\delta$ the righthand side of* (U) *for the above ϕ_k and $a_{j,k}$ is absolutely convergent at (t,x) and satisfies* (E), (C).

3. Sketch of the proof of the theorem. The $\phi_k((t)_k, x)$, called multicharacteristic functions, are defined inductively by

$$(\Phi)\quad \begin{cases} (\partial/\partial t_k)\phi_k((t)_k, x)+\lambda_k(t_1+\cdots+t_k, x; \partial\phi_k/\partial x)=0 & (k=1,2,\cdots), \\ \phi_k((t)_{k-1}, 0, x)=\phi_{k-1}((t)_{k-1}, x) & (k=2,3,\cdots), \\ \phi_1(0,x)=x_1, \end{cases}$$

where $\lambda_k=\lambda_\kappa$ if $\kappa=(k-1)(\mathrm{mod}\ m)+1$, $1\leq\kappa\leq m$.

In order to determine $a_{j,k}((t)_k, x)$, we prepare some notions. First we introduce the induced operators $H_\alpha^k=H_\alpha^k(t,x;\partial_x,\partial_{t_k},\cdots,\partial_{t_{k-m+\alpha}})$ and $H_{\alpha,\beta}^k=H_{\alpha,\beta}^k((t)_k, x;\partial_x,\partial_{t_k},\cdots,\partial_{t_{k-m+\alpha}})$ $(0\leq\alpha\leq m,\ 0\leq\beta\leq\alpha,\ k=1,2,\cdots)$ of H defined by the following relations:

$$H(t,x;\partial_t,\partial_x)\int_0^t dt_1\cdots\int_0^{t-t_1-\cdots-t_{k-2}} dt_{k-1} f((t)_k, x)$$
$$=\sum_{\alpha=0}^m \left[\int_0^t dt_1\cdots\int_0^{t-t_1-\cdots-t_{k-\alpha-2}} dt_{k-\alpha-1} H_{m-\alpha}^k f\right]_{t_k=\cdots=t_{k-\alpha+1}=0},$$

$$H_\alpha^k(t,x;\partial_x,\partial_{tk},\cdots,\partial_{t_{k-m+\alpha}}) f_0(\phi_k((t)_k,x) a((t)_k,x)$$
$$=\textstyle\sum_{\beta=0}^\alpha f_{-(\alpha-\beta)}(\phi_k) H_{\alpha,\beta}^k a,$$

where $[\cdots]_{t_\beta=\cdots=t_\alpha=0}$ means the restriction of $t_\beta,\cdots,t_\alpha$ to 0 if $\beta\geq\alpha$ (and no constraint is made otherwise), and f and a are any holomorphic functions on the chain of the integral. We define similarly I_α^{lk} and $I_{\alpha,\beta}^{lk}$ from $I^l=\partial_t^l$.

Next we introduce the 0-*th* transport equation. Substitute (U) in (E) and (C), arrange the left hands of (E) and (C) with respect to $f_j(\phi_k((t)_k, x))$, and determine the coefficients $a_{j,k}$ so that (U) may be a formal solution of (E), (C). Then the 0-*th* transport equation is given by

$$(\mathrm{T})^0_{j,k}\quad \begin{cases} [a_{j,k}((t)_k, x)]_{t_k=\cdots=t_1=0} \\ \qquad + \sum_{\alpha=0}^{k-1} [\sum_{\beta=0}^{k-\alpha} I^{k-1}_{k-\alpha-1,k-\alpha-\beta-1} a_{j+\beta,\alpha}((t)_\alpha, x)]_{t_\alpha=\cdots=t_1=0} \\ \qquad = \delta_{-p_{k-1},j} c_{k-1}(x') \qquad \text{for } k=1,\cdots,m \text{ and } j\in Z; \end{cases}$$

$$(\mathrm{T})^0_{j,k}\quad \begin{cases} [a_{j,k}((t)_k, x)]_{t_k=\cdots=t_{k-m+1}=0} \\ \quad + \sum_{\alpha=0}^{m-1} [\sum_{\beta=0}^{m-\alpha} H^{k-m+\alpha}_{m-\alpha,m-\alpha-\beta} a_{j+\beta,k-m+\alpha}((t)_{k-m+\alpha}, x)]_{t_{k-m+\alpha}=\cdots=t_{k-m+1}=0} \\ \quad = 0 \qquad \text{for } k=m+1, m+2, \cdots \text{ and } j\in Z. \end{cases}$$

We regard this equation as the one giving information about $[a_{j,k}((t)_k, x)]_{t_k=\cdots=t_{\max(1,k-m+1)}=0}$.

Finally we introduce the μ-*th* transport equation ($\mu=1,\cdots,\min(k,m)$) as follows:

$$(\mathrm{T})^{m-\mu}_{j,k}\quad \begin{cases} \left(\dfrac{\partial}{\partial t_{k-\mu}} + \left[\nabla_\zeta \lambda_{k-\mu}\left(t, x; \dfrac{\partial\phi k}{\partial x}\right)\right]_{t_k=\cdots=t_{k-\mu+1}=0} \cdot \nabla_x\right)[a_{j,k}((t)_k, x)]_{t_k} \\ \quad {}_{=\cdots=t_{k-\mu+1}=0} + \sum_{|\gamma|=2}^{m-\mu} \dfrac{1}{\gamma!}\left[\left(\dfrac{\partial}{\partial\zeta}\right)^\gamma \lambda_{k-\mu}\left(t, x; \dfrac{\partial\phi k}{\partial x}\right)\right]_{t_k} \\ \quad {}_{=\cdots=t_{k-\mu+1}=0}\left(\dfrac{\partial}{\partial x}\right)^\gamma [a_{j-\gamma+1,k}((t)_k, x)]_{t_k=\cdots=t_{k-\mu+1}=0} = 0 \\ \qquad \text{for } \mu=0,1,\cdots,\min(k-1, m-1). \end{cases}$$

Now the $a_{j,k}((t)_k, x)$ ($j\in Z, k\in N$) can be determined by solving the equations $(\mathrm{T})^\mu_{j,k}$ ($\mu=0,\cdots,m$), which we call simply the transport equations.

Estimating these ϕ_k and $a_{j,k}$, we can obtain the theorem.

These results will be published elsewhere with a detailed proof.

References

[1] Hamada, Y.: The singularities of the solutions of the Cauchy problem. Publ. RIMS, Kyoto Univ., **5**, 21–40 (1969).

[2] Hamada, Y., Leray, J., and Wagschal, C.: Systèmes d'équations aux dérivées partielles a caractéristiques multiples problème de Cauchy ramifié; hyperbolicité partielle. J. Math. Pures Appl., **55**, 297–352 (1976).

[3] Kumano-go, H., and Taniguchi, K.: Fourier integral operaotrs of multiphase and the fundamental solution for a hyperbolic system. Funkcial. Ekvac. (to appear).

[4] Nakamura, G.: The singularities of solutions of the Cauchy problems for systems whose characteristic roots are non-uniform multiple. Publ. RIMS, Kyoto Univ., **13**, 255–275 (1977/78).

[5] Hamada, Y., and Nakamura, G.: On the singularities of the solution of the Cauchy problem for the operator with no uniform multiple charactreistics. Ann. Scuola Norm. Sup. Pisa, **4**, 725–755 (1977) .