11. On 2p-Fold Transitive Permutation Groups. II

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§0. **Introduction.** The purpose of this paper is to extend the results of Yoshizawa [7] and Bannai [2].

In §§1 and 2, we shall prove the following results which are obtained by improving some parts of the proofs of [7].

Theorem 1. *Let p be an odd prime $\geqslant 11$, and let q be an odd prime with $p<q\leqslant p+\frac{p-1}{2}$. Let G be a $2p$-fold transitive permutation group on a set $\Omega=\{1,2,\cdots,n\}$. If the order of $G_{1,2,\ldots,2p}$ is not divisible by q, then G is S_n $(2p\leqslant n\leqslant 2p+q-1)$ or A_n $(2p+2\leqslant n\leqslant 2p+q-1)$.*

Theorem 2. *Let p be an odd prime $\geqslant 11$, and let q be an odd prime with $p<q\leqslant p+\frac{p-1}{2}$. Let G be a $2p$-fold transitive permutation group on a set $\Omega=\{1,2,\cdots,n\}$. If $G_{1,2,\ldots,2p}$ has an orbit on $\Omega-\{1,2,\cdots,2p\}$ whose length is less than q, then G is S_n $(2p+1\leqslant n\leqslant 2p+q-1)$ or A_n $(2p+2\leqslant n\leqslant 2p+q-1)$.*

As an immediate corollary to Theorem 2, we have the following

Corollary. *Let p be an odd prime $\geqslant 11$, and let q be an odd prime with $p<q\leqslant p+\frac{p-1}{2}$. Let D be a $2p$-$(v,k,1)$ design with $2p<k<2p+q$. If an automorphism group G of D is $2p$-fold transitive on the set of points of D, then D is a $2p$-$(k,k,1)$ design, namely a trivial design.*

In §3, by making use of the above results and the proof of [2, Theorem A], we shall prove the following

Theorem 3. *Let p be an odd prime $\geqslant 11$, and let q be an odd prime with $p<q\leqslant p+\frac{p-1}{2}$. Let G be a $2p$-fold transitive permutation group on a set $\Omega=\{1,2,\cdots,n\}$. If a Sylow q-subgroup Q of $G_{1,2,\ldots,2p}$ is semiregular on $\Omega-I(Q)$, then G is S_n $(2p\leqslant n\leqslant 2p+2q-1)$ or A_n $(2p+2\leqslant n\leqslant 2p+2q-1)$.*

§1. **Proof of Theorem 1. Lemma 1.** *Let p be a prime $\geqslant 11$ and q be a prime with $p<q\leqslant p+\frac{p-1}{2}$, and let r be an integer with*

$0 \leqslant r \leqslant q-p-1$. *Then there exists no permutation group G on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following condition: For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of Ω, $p \mid |G_{\alpha_1,\cdots,\alpha_{2p}}|$ and $q \nmid |G_{\alpha_1,\cdots,\alpha_{2p}}|$ hold, and any element of order p fixing at least $2p$ points fixes $2p+r$ or $3p+r$ points, and moreover G contains an element of order p fixing $3p+r$ points.*

Proof. Let G be a group satisfying the above condition, and let a be an element of G of order p fixing $3p+r$ points. We may assume that $a=(1)\cdots(3p+r)(3p+r+1, \cdots, 4p+r)\cdots$. Set $T=C_G(a)^{I(a)}_{3p+r+1,\cdots,4p+r}$. Then T is a permutation group on $I(a)$ ($|I(a)|=3p+r$) satisfying the following: For any p points $\alpha_1, \cdots, \alpha_p$ of $I(a)$, $p \mid |T_{\alpha_1,\cdots,\alpha_p}|$ and $q \nmid |T_{\alpha_1,\cdots,\alpha_p}|$ hold. We will show that such T does not exist. Let $\Delta_1, \cdots, \Delta_s$ be the orbits of T with $|\Delta_i| \geqslant p$ ($i=1, \cdots, s$), where s is at most three because of $|I(a)|<4p$. Suppose that $s=3$. Set $|\Delta_i|=p+k_i$ ($i=1, 2, 3$). Then $(p+k_1)+(p+k_2)+(p+k_3) \leqslant 3p+r$. Hence, $(k_1+1)+(k_2+1)+(k_3+1) \leqslant r+3<p$, which contradicts the property of T. Suppose that $s=2$. We may assume that $|\Delta_1| \geqslant |\Delta_2|$. Set $|\Delta_2|=p+k$. We divide the consideration into the following two cases: (I) $p+k \geqslant q$, (II) $p+k<q$. First assume that the case (I) holds. Since $3p+r \leqslant (2p-1)+q$ and $|\Delta_2|=p+k \geqslant q$, we have $|\Delta_1|<2p$. Then for p points $\alpha_1, \cdots, \alpha_p$ of Δ_1, $T_{\alpha_1,\cdots,\alpha_p}$ contains a p-cycle $(\beta_1, \cdots, \beta_p)$ with $\{\beta_1, \cdots, \beta_p\} \subseteq \Delta_2$. Set $H=\langle(\beta_1, \cdots, \beta_p)^x \mid x \in T^{\Delta_2}\rangle$. Then H^{Δ_2} is transitive, because T^{Δ_2} is transitive and $|\Delta_2| \leqslant 2p-1$. Furthermore, H^{Δ_2} is primitive, and so H^{Δ_2} is $(|\Delta_2|-p+1)$-transitive by [5, Theorem 13.8]. On the other hand, H is a subgroup of T with $H^{\Delta_1}=1$. Hence, since H contains a q-cycle, T contains the q-cycle. This is a contradiction. Next assume that the case (II) holds. Let $\beta_1, \cdots, \beta_{k+1}$ be $k+1$ points of Δ_2. Then the stabilizer in $T^{\Delta_1}_{\beta_1,\cdots,\beta_{k+1}}$ of $p-k-1$ points contains an element of order p. In this case, since $k<q-p \leqslant \frac{p-1}{2}$, we have $p-k-1 \geqslant \frac{p+1}{2} \geqslant 6$. So, if T^{Δ_1} is primitive, then we have $T^{\Delta_1} \geqslant A^{\Delta_1}$ by [5, Theorem 13.10]. In particular, since $|\Delta_1| \geqslant p+(p-k-1) \geqslant p+\frac{p+1}{2}>q$, T^{Δ_1} contains a q-cycle. Therefore T contains the q-cycle, a contradiction. Thus, T^{Δ_1} is imprimitive. If the length of a block of T^{Δ_1} is at least p, then we get a contradiction by a similar argument to the case $s=3$. Therefore, the length of any block of T^{Δ_1} is less than p. Hence we have $|\Delta_1| \geqslant 2(p-k-1)+2p \geqslant 3p+1$, a contradiction. Suppose that $s=1$. First assume that T^{Δ_1} is imprimitive. If the length of a block of T^{Δ_1} is at least p, then we get a contradiction by similar arguments to the cases $s=3$ and $s=2$. Therefore, the length of any block of T^{Δ_1} is less than p. Hence we have $|\Delta_1| \geqslant 2p+2p=4p$, a contradiction. Thus T^{Δ_1} is primitive. Since the stabilizer in T^{Δ_1} of p points contains an element of order p, we have $T^{\Delta_1} \geqslant A^{\Delta_1}$ by [5, Theorem 13.10]. Hence T contains a q-cycle, a contradiction.

Proof of Theorem 1. Let G be a counter example to the theorem. Let P be a Sylow p-subgroup of $G_{1,2,\dots,2p}$. Then $P\neq 1$ and P is not semiregular on $\Omega-I(P)$, by [1, Main Theorem] and [2, Theorem 1]. Set $|I(P)|=2p+r$, where $0\leqslant r\leqslant p-1$. First assume that $r\geqslant q-p$. Let R be a subgroup of P such that the order of R is maximal among all subgroups of P fixing at least $3p$ points. By Theorems A and B in [6], we have $|I(R)|=3p+r$ $(\geqslant 2p+q)$ and there exist $2p$ points $\alpha_1,\cdots,\alpha_{2p}$ in $I(R)$ such that $N_G(R)^{I(R)}_{\alpha_1,\dots,\alpha_{2p}}$ contains a q-cycle, a contradiction. Next assume that $r\leqslant q-p-1$. Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing at least $4p$ points. Set $N=N_G(Q)^{I(Q)}$. Then N is a permutation group on $I(Q)$ $(|I(Q)|\geqslant 4p)$ satisfying the following: For any $2p$ points $\alpha_1,\cdots,\alpha_{2p}$ of $I(Q)$, $p\,|\,|N_{\alpha_1,\dots,\alpha_{2p}}|$ and $q\nmid|N_{\alpha_1,\dots,\alpha_{2p}}|$ hold, and any element of order p fixing at least $2p$ points fixes $2p+r$ or $3p+r$ points. Moreover, by Therems A and B in [6], N contains an element of order p fixing $3p+r$ points. Hence, we get a contradiction by Lemma 1.

§2. Proof of Theorem 2. Lemma 2. *Let p be a prime and $t=p+2$, and k be an integer with $t+1$ $(=p+3)\leqslant k\leqslant t+p-1$ $(=2p+1)$. Then there exists no permutation group G on a set $\Omega=\{1,2,\cdots,n\}$ which satisfies the following condition: For any t points $\alpha_1,\cdots,\alpha_t$ of Ω, a Sylow p-subgroup P of $G_{\alpha_1,\dots,\alpha_t}$ is nontrivial and semiregular on $\Omega-I(P)$, where $I(P)$ depends on $\{\alpha_1,\cdots,\alpha_t\}$ but not on the choice of P, and $|I(P)|=k$.*

Proof. Let G be a group satisfying the above condition, and let a be an element of G of order p fixing k points. We may assume that $a=(1)\cdots(t)\cdots(k)(k+1,\cdots,k+p)\cdots$. Set $T=C_G(a)^{I(a)}_{k+1,\dots,k+p}$. Then T is a permutation group on $I(a)$ $(|I(a)|=k\geqslant p+3)$ satisfying the following: For any two points α,β of $I(a)$, a Sylow p-subgroup S of $T_{\alpha\beta}$ is a cyclic group generated by a p-cycle, and $I(S)$ depends on $\{\alpha,\beta\}$ but not on the choice of S. We will show that such T does not exist. We may assume that T contains a p-cycle x of the form $x=(1,2,\cdots,p)$. Let y be a p-cycle in $T_{1,p+1}$. Since $x,y\in T_{p+1}$ and $I(x)\neq I(y)$, we have $I(x)\cap I(y)=\{p+1\}$. So, $\{p+2,\cdots,k\}\cap I(y)=\phi$. Hence y fixes a point i with $2\leqslant i\leqslant p$. Set $w=(x^{i-1})^{-1}yx^{i-1}$. Then since $I(y)\cap I(w)\ni i,\ p+1$, we have $I(y)=I(w)$. Assume that y moves some point γ of $\{1,\cdots,p\}$. Then w moves $\gamma^{x^{i-1}}$, and so y moves $\gamma^{x^{i-1}}$. Repeating the argument, y moves $\gamma^{x^{2(i-1)}},\gamma^{x^{3(i-1)}},\cdots$. Hence y moves all points of $\{1,\cdots,p\}$, a contradiction. Thus, we may assume that $y=(p+2,\cdots,2p+1)$, and $k=2p+1$. Let z be a p-cycle in $T_{1,p+2}$. Then $|I(x)\cap I(z)|\geqslant 2$ or $|I(y)\cap I(z)|\geqslant 2$ holds. This is a contradiction, because $I(x)\neq I(z)$ and $I(y)\neq I(z)$.

Proof of Theorem 2. Let G be a counter example to the theorem.

Let Δ be an orbit of $G_{1,2,\cdots,2p}$ on $\Omega-\{1,2,\cdots,2p\}$ such that $|\Delta|<q$. By [4, IV], we have $2\leqslant|\Delta|<q$. Let Q be a Sylow q-subgroup of $G_{1,2,\cdots,2p}$. Then by Theorem 1 and the lemma of Witt [5, Theorem 9.4], we have $Q\neq 1$ and $N_G(Q)^{I(Q)}\geqslant A^{I(Q)}$. Since $I(Q)\supset\Delta$ and $N_G(Q)_{1,\cdots,2p}^{I(Q)-\{1,\cdots,2p\}}\geqslant A^{I(Q)-\{1,\cdots,2p\}}$, we have $I(Q)=\Delta\cup\{1,2,\cdots,2p\}$. This shows that $I(Q)$ depends on $\{1,2,\cdots,2p\}$ but not on the choice of Q. Let R be a subgroup of Q such that the order of R is maximal among all subgroups of Q fixing more than $|I(Q)|$ points. Set $N=N_G(R)^{I(R)}$. Then N is a permutation group on $I(R)$ ($|I(R)|\geqslant 2p+q$) satisfying the following: For any $2p$ points $\alpha_1,\cdots,\alpha_{2p}$ of $I(R)$, a Sylow q-subgroup S of $N_{\alpha_1,\cdots,\alpha_{2p}}$ is nontrivial and semiregular on $I(R)-I(S)$, where $I(S)$ depends on $\{\alpha_1,\cdots,\alpha_{2p}\}$ but not on the choice of S, and $|I(S)|=2p+|\Delta|\leqslant 2p+q-1$. Hence, we get a contradiction by Lemma 2.

§3. Proof of Theorem 3. Lemma 3. *Let G be a 4-transitive but not 5-transitive group on a set $\Omega=\{1,2,\cdots,n\}$. If $G_{\{1,2,3,4\}}$ does not stabilize any orbit of G_{1234} as a set, then G is A_6.*

Proof. Let G be a counter example to the lemma. Let P be a Sylow 2-subgroup of G_{1234}. Then by [3] and [4, VII, VIII, IX, XI], P is nontrivial and is not semiregular on $\Omega-I(P)$, and $|I(P)|=4$ or 5. If $|I(P)|=5$, then G_{1234} has a unique orbit Δ on $\Omega-\{1,2,3,4\}$ such that $|\Delta|$ is odd. So, $G_{\{1,2,3,4\}}$ stabilizes Δ as a set, a contradiction. Hence, we have $|I(P)|=4$. Let R be a subgroup of P such that the order of R is maximal among all subgroups of P fixing at least six points. Then by Theorem 1 in [4, X], $N_G(R)^{I(R)}=S_6$, A_8 or M_{12}. We may assume that $I(R)\ni 5,6$. Let Δ be the orbit of G_{1234} such that $\Delta\ni 5,6$. By the assumption on G, $G_{\{1,2,3,4\}}$ contains an element g such that $\Delta\neq\Delta^g$ (where Δ^g is an orbit of G_{1234}). Since S_6, A_8 and M_{12} are 5-transitive groups, $N_G(R)$ contains an element h such that $g^{\{1,2,3,4\}}=h^{\{1,2,3,4\}}$ and $5^h=6$. Then we have $\Delta^{h^{-1}}=\Delta$. Hence, we have $\Delta^{h^{-1}g}=\Delta^g\neq\Delta$. Since $h^{-1}g\in G_{1234}$, this is a contradiction.

Proof of Theorem 3. Let G be a counter example to the theorem. Let Q be a Sylow q-subgroup of $G_{1,2,\cdots,2p}$. Then Q is semiregular, and $Q\neq 1$ by Theorem 1. Furthermore, by [2, Theorem A], we have $|\Omega-I(Q)|\equiv q \pmod{q^2}$. We may assume that Q contains an element a of order q such that $a=(1)\cdots(2p+r)(2p+r+1,\cdots,2p+r+q)\cdots$, where $0\leqslant r\leqslant q-1$. Since $n-(2p+r)\equiv q \pmod{q^2}$, G contains an element of order q which fixes less than $2p+r$ points. Hence by the proof of Theorem A in [2], we see that $C_G(a)_{1,\cdots,2p-2,\{2p-1,2p\},2p+1,\cdots,2p+r}$ is transitive on $\Omega-\{1,\cdots,2p+r\}$ (cf. [2, (1.4) in §1]). Hence by Theorem 2, $G_{1,\cdots,2p-2,\{2p-1,2p\}}$ is transitive on $\Omega-\{1,\cdots,2p\}$, because of $r\leqslant q-1$. Hence by Lemma 3, $G_{1,\cdots,2p}$ is transitive on $\Omega-\{1,\cdots,2p\}$. Thus G is $(2p+1)$-transitive on Ω. Repeating the argument, we infer that G is $(2p+r+1)$-transitive on Ω. Since $q \nmid |G_{1,\dots,2p+r+1}|$, we have $G \geqslant A^{\Omega}$ by Theorem 1, a contradiction.

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