By Michio Jimbo and Tetsuji Miwa

Research Institute for Mathematical Sciences,
Kyoto University

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We report the following two results on the diagonal spin-spin correlation function \( \langle \sigma_0 \sigma_{NN} \rangle \) of the two dimensional Ising lattice. (i) \( \langle \sigma_0 \sigma_{NN} \rangle \) satisfies a non-linear ordinary differential equation with respect to the temperature, which is equivalent to a sixth Painlevé equation (P VI). (ii) \( \langle \sigma_0 \sigma_{NN} \rangle \) satisfies a non-linear ordinary difference equation with respect to \( N \). In the scaling limit, both the differential equation (i) and the difference equation (ii) reduce to known results [1] related to P V (or an equivalent of it, P III [2]) on the scaled two point function.

Our method is to construct an isomonodromy family of linear differential equations in such a way that its \( \tau \) function [3] coincides with \( \langle \sigma_0 \sigma_{NN} \rangle \). The difference equation (ii) is a consequence of the relations among the \( \tau \) function and its Schlesinger transforms [4], [5].

Recently McCoy-Wu [6] and Perk [7] have obtained difference equations for \( \langle \sigma_0 \sigma_{NN} \rangle \). The relations between their works and ours (for \( M=N \)) is yet to be clarified.

1. Results. We follow the notations of [8], [9]. Let \( \langle \sigma_0 \sigma_{NN} \rangle_{T<T_c} \) (resp. \( \langle \sigma_0 \sigma_{NN} \rangle_{T>T_c} \)) denote the diagonal spin-spin correlation function below (resp. above) the critical temperature, where we use the parametrization

\[
T = \frac{1}{2} \log \left( \frac{\sinh \beta E_1 + \sinh \beta E_2}{\sinh \beta E_1 - \sinh \beta E_2} \right),
\]

with \( T > 1 \), \( \beta = 1/kT \). We set

\[
\sigma_{NN}^\pm(t) = (t(t-1)) \frac{d}{dt} \log \langle \sigma_0 \sigma_{NN} \rangle_{T<T_c} \pm \frac{1}{4},
\]

\[
\sigma_{NN}^\pm(t) = (t(t-1)) \frac{d}{dt} \log \langle \sigma_0 \sigma_{NN} \rangle_{T>T_c} \pm \frac{1}{4}.
\]

Then both \( \tau = \sigma_{NN} \) are solutions of the following second order non-linear ordinary differential equation.

\[
\left( t(t-1) \frac{d^2 \sigma}{dt^2} \right)^2 = N^2 \left( t-1 \frac{d \sigma}{dt} - \sigma \right)^2 - \frac{d \sigma}{dt} \left( t-1 \frac{d \sigma}{dt} - \sigma - \frac{1}{2} \right) \left( t+1 \frac{d \sigma}{dt} - \sigma \right).
\]

The equation (3) is equivalent to the sixth Painlevé equation (5.55) [4].
with parameters \( \alpha = (N - 3/2)^2 / 2 \), \( \beta = -(N + 1/2)^2 / 2 \), \( \gamma = 1/8 \), \( \delta = 3/8 \).

The difference equations for \( \langle \sigma_0 \sigma_{N,N} \rangle_{T+\geq T_e} \) are written as a first order system. We introduce a set of dependent variables \( \alpha_N, \beta_N, \) etc. tabulated below.

\[
G_N^{(p)} = \frac{1}{\alpha_N} \left( \begin{array}{c} \alpha_{N-1} \\ \beta_N \\ \alpha_{N+1} \end{array} \right), \quad G_N^{(p)} = \frac{1}{\alpha_N} \left( \begin{array}{c} \alpha_{N} \\ \beta_{N} \\ \alpha_{N} \end{array} \right)
\]

where \( \det G_N^{(p)} = 1 \), \( \det G_N^{(p)} = 1 \). These quantities \( (4) \) satisfy the following bilinear difference equations:

\[
\begin{align*}
(2N + 1) \alpha_N \alpha_{N-1} - (2N - 1) \alpha_N - \alpha_{N-1} \alpha_{N+1} - \alpha_{N-2} \alpha_{N+2} &= 0, \\
\beta_N \beta_{N-1} - \beta_N \beta_{N+1} - \beta_{N-2} \beta_{N+2} &= 0, \\
\gamma_N \gamma_{N-1} - \gamma_N \gamma_{N+1} - \gamma_{N-2} \gamma_{N+2} &= 0,
\end{align*}
\]

The correlation functions are related to \( (4) \) through

\[
\psi_{\pm, N} = \frac{1}{\alpha_N} \psi_{\pm, N-1} \psi_{\pm, N+1}
\]

where \( \alpha_N, \gamma_N \) correspond to the solution of \( (5) \) with the initial condition

\[
\psi_{\pm, 0} = \delta_{\pm, 0} = 0, \quad \psi_{\pm, 1} = -t^{1/4} (t - 1)^{1/4} \gamma_{\pm, 0} = 0
\]

The correlation functions are related to \( (4) \) through

\[
\langle \sigma_0 \sigma_{N,N} \rangle_{T+\geq T_e} = -t^{-1/2} (t - 1)^{-1/2} \psi_{\pm, 0}
\]

where \( \alpha_N, \gamma_N \) correspond to the solution of \( (5) \) with the initial condition

\[
\psi_{\pm, 0} = \delta_{\pm, 0} = 0, \quad \psi_{\pm, 1} = -t^{1/4} (t - 1)^{-1/4},
\]

The commutator product of the free field \( \psi_{\pm} \) and the "spin operator" \( \varphi_N \) is given by

\[
[\psi_{\pm}, \varphi_N] = 2\sqrt{\omega} z^{-N} \varphi_N(z)
\]

where \( \varphi_N(z) = : \phi_N(z) \exp(\sigma_N/2) : \) and \( \phi_N(z) = \int \frac{d\theta}{2\pi} \sqrt{z} \varphi_N(z) \frac{\omega^{1/2} \varphi_n(\theta)}{z^{1/2} - z} \)

We also set \( \phi_N = \phi_N + (0) = \int \frac{d\theta}{2\pi} \omega^{1/2} \varphi_N(\theta) z^{1/2} \) and \( \varphi_N = \varphi_N + (0) = \phi_N \exp(\sigma_N/2) : \)

The correlation function \( \langle \sigma_0 \sigma_{N,N} \rangle_{T+\geq T_e} \) is given by
With $a = \sinh \beta E_1 \sinh \beta_1 E_1 = 1/\sqrt{t}$.

3. Construction of an isomonodromy family. We define a $2 \times 2$ matrix $Y(z, z_0) = \hat{Y}(z, z_0)\left(\begin{smallmatrix} \omega & -N \\ -1 & 1 \end{smallmatrix}\right)$ by the following series.

\[
\hat{Y}(z, z_0) = 1 + \frac{\left(-\frac{z}{z_0}\right)^{t-1}}{2\pi i} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_{t-1}}{2\pi} \frac{z - z_0}{z_1 - z_0} f_1(z_1, z_2)f_2(z_1, z_3) \cdots f_N(z_1, z_t),
\]

where $f_j(z, z') = \frac{\omega}{z^{N+1}}\left(1 - e^{-t(z - z')\pm i\theta}\right)$ with $\kappa = +$ or $-$. $Y(z, z_0)$ is so normalized that $\hat{Y}(z, z_0) = 1 + O(z - z_0)$ ($z_0 \neq \infty$) or $1 + O(1/z)$ ($z_0 = \infty$). Moreover we have $\det \hat{Y}(z, z_0) = 1$.

We denote by $\hat{Y}_+(z, z_0)$ the restriction of $\hat{Y}(z, z_0)$ to $D_\pm = \{z \mid |z| \leq 1\}$, and set $Y_+(z, z_0) = \hat{Y}_+(z, z_0)\left(\begin{smallmatrix} \omega & -N \\ -1 & 1 \end{smallmatrix}\right)$. The connection between $Y_+(z, z_0)$ is given by

\[
Y_{-}(z, z_0) = Y_{+}(z, z_0) \left(\begin{array}{cc} 1 - \lambda^2 & -\lambda \\ \lambda & 1 \end{array}\right) (\kappa = +),
\]

\[
Y_{-}(z, z_0) = Y_{+}(z, z_0) \left(\begin{array}{cc} 1 - \lambda^2 & \lambda \\ \lambda & 1 \end{array}\right) (\kappa = -).
\]

If we modify the expectation value so that $\langle \psi(\theta)\psi^*(\theta') \rangle = 2\pi \delta(\theta - \theta')$, we obtain the following identities.

\[
\hat{Y}_+(z, z_0) = 1 + \left(1 - \frac{z}{z_0}\right) \frac{\langle \phi_{-N}(z)\phi^*_N(z_0) \rangle \exp (\rho/N) \exp (\rho/N)}{\langle \phi_0 \phi_{-N} \rangle},
\]

if $N \leq 0$, or $N = 0$ and $|z_0| \geq 1$, or $N = 0$ and $\varepsilon = \pm$ (for $\kappa = \pm$).

\[
\hat{Y}_+(z, z_0) = \left(1 - \frac{z}{z_0}\right) \frac{\langle \phi^*_N(z)\phi^*_N(z_0) \rangle}{\langle \phi_0 \phi_{N} \rangle},
\]

if $N \leq 0$, or $N = 0$ and $|z_0| \geq 1$, or $N = 0$ and $\varepsilon = \mp$ (for $\kappa = \pm$).

\[
\hat{Y}_+(z, z_0) = \left(1 - \frac{z}{z_0}\right) \frac{\langle \phi^*_0(z_0)\phi^*_N(z) \rangle}{\langle \phi_0 \phi_{N} \rangle},
\]

if $N \leq 0$, or $N = 0$ and $|z_0| \leq 1$, or $N = 0$ and $\varepsilon = \pm$ (for $\kappa = \pm$).

\[
\hat{Y}_+(0, \infty) = \frac{\langle \phi_0 \phi_{-N} \rangle}{\langle \phi_0 \phi_{N} \rangle} \quad \text{if $\kappa = \pm$, $N \leq 0$},
\]

\[
\hat{Y}_+(0, \infty) = \frac{\langle \phi^*_0 \phi^*_N \rangle}{\langle \phi^*_0 \phi^*_N \rangle} \quad \text{if $\kappa = \pm$, $N \leq 0$},
\]

\[
\hat{Y}_+(0, \infty) = \frac{\langle \phi^*_0 \phi^*_N \rangle}{\langle \phi^*_0 \phi^*_N \rangle} \quad \text{if $\kappa = \pm$, $N \leq 0$}.
\]

From (8), (14) and (15) we obtain
4. Deformation and the Schlesinger transformation. The construction in § 3 entails the following monodromy property for the matrix $Y(z)=Y_{z}(z, \infty)$. It is a multi-valued analytic matrix with four regular singularities at $z=0, a, a^{-1}$ and $\infty$, where the local exponents are given by

$$\begin{align*}
T^{(0)}_{-0} &= \begin{pmatrix} -N - \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, & T^{(+)}_{-0} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, & T^{(-)}_{-0} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, & T^{(\infty)}_{-0} &= \begin{pmatrix} N - \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix},
\end{align*}$$

respectively. Moreover its global monodromy matrices $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ at $z = a^{-1}, \infty$ and $\begin{pmatrix} -1 + 2\lambda & 2\lambda \\ 2\lambda(1 - \lambda) & 1 - 2\lambda \end{pmatrix}$ (if $\kappa = +$), $\begin{pmatrix} -1 + 2\lambda & 2\lambda(1 - \lambda) \\ 2\lambda & 1 - 2\lambda \end{pmatrix}$ (if $\kappa = -$) at $z = 0, a$ are independent of $a^{\pm1}$. These properties are sufficient to guarantee that $Y_{N}(z)$ should satisfy linear differential equations of the form (cf. [3], [5])

$$\begin{align*}
(26) & \quad \frac{\partial Y_{N}}{\partial z} = \left( -\frac{A_{+}}{z} + \frac{A_{-}}{z-a} + \frac{1}{a} \right) Y_{N} \\
& \quad \frac{\partial Y_{N}}{\partial a} = \left( -\frac{A_{+}}{z-a} + \frac{1}{a} \right) Y_{N} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{align*}$$

Here we have set

$$\begin{align*}
A_{\pm} &= G^{(\pm)}_{N} T^{(\pm)}_{\pm} G^{(\pm)}_{N}^{-1} (\nu = 0, \pm), & A_{a} + A_{a^{-1}} &= -T^{(\infty)}_{\pm}.
\end{align*}$$

By a change of variables $z=ax, t=a^{-i}, Z(x) = KY_{N}(ax), K = \begin{pmatrix} i a^{N-1} \\ 1 \end{pmatrix}$, the integrability condition for (26) reduces to a sixth Painlevé equation (5.55) in [4] with parameters $\alpha = (N-3/2)/2, \beta = -(N+1/2)/2, \gamma = 1/8, \delta = 3/8$. Correlation functions are related to the $\tau$ function $\tau_{N}(t)$ associated with (26); by comparing (24) with the defining equation $d \log \tau_{N}(t) = \text{trace} \left( A_{a} A_{a^{-1}} (da/a) + A_{a} A_{a^{-1}} (da^{-1}/a^{-1}) + A_{a} A_{a^{-1}} (da^{-1}/a^{-1}) \right)$, we find

$$\begin{align*}
(29) & \quad \langle \phi \sigma_{N} \rangle = \text{const. } t^{1/2}(t-1)^{-1/4} \tau_{N}(t),
\end{align*}$$

The result (3) for $\langle \phi \sigma_{N} \rangle_{T_{-}<\tau_{s}}$ follows from (8), (29) and (5.60) [4].

To derive difference equations, we observe that changing $N$ into $N-1$ amounts to shifting the exponents by integers (Schlesinger transformation) as $T^{(0)}_{\pm} \rightarrow T^{(0)}_{\pm} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, T^{(\infty)}_{\pm} \rightarrow T^{(\infty)}_{\pm} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. It is known [4] [5] that such transformations are achieved by multiplication by a rational matrix $R_{N}(z)$:
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(30) \[ Y_{N-1}(z) = R_N(z)Y_N(z), \quad R_N(z) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}z + R_{0N} \]

\[ R_{0N} = \begin{pmatrix} (Y_{2N}^{(\omega)})_{22}(G_{21}^{(\omega)})_{21} - (Y_{2N}^{(\omega)})_{12} \\ 1 \end{pmatrix} = G_{N-1}^{(\omega)}(0 \quad 1)G_{N-1}^{(\omega) -1}. \]

Here \( Y_{2N}^{(\omega)} \) signifies the coefficient matrix of \( \hat{Y}_y(z) = 1 + Y_{2N}^{(\omega)}z^{-1} + \cdots \) \( (z \to \infty) \). In particular, (30) implies

(31) \[ G_{N+1}^{(\omega)} = R_N(a_{\omega})G_{N}^{(\omega)}. \]

If we write down (31) and the constraint (27) in terms of the parameters \( \alpha_N, \beta_N, \cdots \) given in (4), we obtain (5). However, care must be taken in identifying \( \alpha_N, \gamma_N \) with \( \langle \varphi_0 \varphi_N \rangle \) and \( \langle \varphi_0 \varphi_N \rangle \). As is shown in (20)-(23), the latter correspond to different monodromy problems \( \epsilon = + \) or \( \epsilon = - \) according to the sign of \( N \), and hence to different solutions of (5). This explains the appearance of \( |N| \) in (6). Finally the differential equation (3) for \( \langle \varphi_0 \varphi_N \rangle_{T_1} \) is obtained by noting that \( \langle \varphi_0 \varphi_N \rangle \) coincides with the \( \tau \) function corresponding to the Schlesinger transformation \( T_{\omega}^{(\omega)} \to T_{\omega}^{(\omega)} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) (Theorem 4.1[4]).

5. Scaling limit. Here we shall show that the previously known results [1] are reproduced from (3), (5) in the scaling limit \( N \to \infty, t = 1 + N^{-1}t \) with \( t > 0 \) fixed.

In this limit the confluence of two regular singularities \( z = 0, \infty \) takes place to produce an irregular singularity of rank 1.

Since the monodromy stays constant as we vary \( N \), the limiting monodromy data are determined from the original ones. To see this we scale the infinite series (14) by setting \( z = e^{i\epsilon N}, \; \epsilon = t/2mN \) \( (m > 0, \text{arbitrary}) \). Choosing \( \epsilon = + \) we then have

(32) \[ \lim_{\epsilon \to 0} \left( \epsilon^{-1} 1 \right) Y_{z-N}(z, \infty) = \hat{Y}_{z}(p) = \hat{Y}_{z}(p)e^{i\epsilon N \epsilon}, \]

where \( \hat{Y}_{z}(p) = \sqrt{p^2 + m^2} \) and \( \hat{Y}_{z}(p) = 1 + O(p^{-1}) \) as \( p \to \infty \) in the region \( \mathcal{D}_z \) (Fig. 1).

![Fig. 1](image)

Modifying (14) slightly we get also, after scaling, \( \hat{Y}_{z}(p) \) which has a similar property in the region \( \mathcal{D}_z \). These are connected through

(33) \[ \hat{Y}_{+}(p) = \hat{Y}_{z}(p)\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \hat{Y}_{-}(p) = \hat{Y}_{z}(p)\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}. \]

The linear differential equations (26) tend to

(34) \[ \frac{\partial \hat{Y}}{\partial p} = \begin{pmatrix} \frac{\partial \hat{A}_+}{p - im} + \frac{\partial \hat{A}_-}{p + im} + \left(i\epsilon/2m\right) \end{pmatrix} \hat{Y}, \frac{\partial \hat{Y}}{\partial \epsilon} = \frac{1}{2m} \begin{pmatrix} -i\epsilon & \bar{b} \\ \bar{b} & 0 \end{pmatrix} \hat{Y}. \]
\( \overline{A}_+ = G^{(+)}(z) T_0^{(+)} G^{(+)}(z)^{-1}, \quad \overline{A}_- = \left( \begin{array}{cc} 1 & -b/2m \\ -\bar{b}/2m & 0 \end{array} \right) \)

\( G^{(+)} = \left( \begin{array}{cc} \bar{a}^{(+)} & \bar{b}^{(+)} \\ \bar{a}^{(+)} & \bar{d}^{(+)} \end{array} \right) = \lim_{\varepsilon \to 0} \left( \begin{array}{c} \varepsilon \\ 1 \end{array} \right) G^{(+)}(\varepsilon^{-1}) \).

Here we have set \( \bar{b} = \lim \varepsilon^{-2}\beta_N/\alpha_N, \quad \bar{c} = \lim \gamma_N/\alpha_N. \)

The difference equations (5) are scaled to give
\[ \frac{d}{dt} \left( G^{(+)}(\varepsilon^{1/2}) \right) = \left( \begin{array}{cc} \mp 1/2 & b/2m \\ -\bar{c}/2m & 0 \end{array} \right) G^{(+)}(\varepsilon^{1/2}). \]

which is one of the equivalent expressions of the deformation equations for (34).

If we set \( \overline{A}_+ = \left( \begin{array}{cc} 1/2 + \bar{z} & -u(1/2 + \bar{z}) \\ \bar{z}/u & -\bar{z} \end{array} \right), \quad \overline{A}_- = \left( \begin{array}{cc} 1/2 - \bar{z} & \bar{u}\bar{y}(-1/2 + \bar{z}) \\ -\bar{z}/\bar{u}y & \bar{z} \end{array} \right), \)

then \( \bar{y} = \bar{y}(t) \) is a solution of PV with \( \alpha = 1/8, \beta = -1/8, \gamma = 0, \delta = -1/2. \)

The relation (29) reduces in the limit to (4.11.9) [1] (with \( \bar{i} = -t \); the factor 1/2 there is erroneous)
\[ \lim \frac{d}{dt} \log \langle \varphi \cdot \varphi \rangle = \left( -\bar{t} \bar{z} + (-2\bar{z}^2 - \bar{y}z_0 \left( \frac{1}{2} - \bar{z} \right) + \bar{y}^{-1}z_0 \left( \frac{1}{2} + \bar{z} \right)) \right), \]

Finally the differential equation (3) is scaled to
\[ \left( \frac{d}{dt} \sigma \left( \sigma - \bar{z} - \frac{d\sigma}{dt} + 2 \frac{d\sigma}{dt} \right) \right)^2 + 4 \left( \frac{d\sigma}{dt} \right)^2 = \left( \frac{1}{4} - \frac{d\sigma}{dt} \right)^2. \]

where \( \sigma(t) = \lim \sigma_{N,t}(t). \)

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References


