55. On F^4-Manifolds and Cell-Like Resolutions

By Masaaki UE
Department of Mathematics, Faculty of Science, University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1980)

1. Statement of results. For a compact metric space X of dimension n, "a cell-like resolution of X" is defined as a pair (M, f), consisting of an n-dimensional topological manifold M and a cell-like map $f: M \rightarrow X$. The purpose of this note is to announce a 4-dimensional version of resolution theorem of generalized manifolds, [1], [2], [5], [6], in terms of F-manifolds whose notion was introduced by Freedman and Quinn [3].

Definition [3]. A topological space M is said to be an *F^4-manifold*, if it is an ENR homology 4-manifold with isolated 1-LC nonmanifold points, disjoint from the boundary. A topological space M is said to be an *F^5-manifold*, if it is an ENR homology 5-manifold whose boundary is collared and is an F^4-manifold, and whose interior is a topological manifold.

The notion of F-manifolds is "a workable substitute" for manifolds in dimension 4. (See [3].)

Main Theorem. *Let X be a 1-connected closed homology 4-manifold, whose nonmanifold set $N(X)$ consists of isolated points. Suppose $X - N(X)$ has a structure of a 1-connected smooth manifold, and $X \times \boldsymbol{R}$ is a 5-dimensional topological manifold. Then there exist a sequence of homology 4-manifolds and cell-like maps between them;*

$$M_0 \xrightarrow{f_0} M_1 \xleftarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xleftarrow{f_3} M_4 = X,$$

where M_i is a homology 4-manifold ($i = 1, \cdots, 4$), M_0 is an F^4-manifold, and f_i is a cell-like map.

The following F^5-version of Quinn's thin h cobordism theorem is essential in proving the main theorem. For the terminology below see [3] and [5].

Theorem A. *Let X be a locally compact metric space, C and D closed subsets of X with $C \supset D$, and ε a positive function on X. Suppose X is locally 1-connected near $C^{2\varepsilon}$, where $C^{2\varepsilon}$ denotes the 2ε-neighborhood of C. Then there exists a positive function $\delta = \delta(X, C, D, \varepsilon)$ on X, having the following property* (*).

(*) *For any compact 1-connected F^5-manifold $(M, \partial_0 M, \partial_1 M)$ with 1-connected boundaries and any proper map $e: M \rightarrow X$, satisfying the condition $(1)_\delta$ below, there are an F^5-manifold $(M', \partial_0 M', \partial_1 M')$ and a proper map $e': M' \rightarrow X$, satisfying the condition* (2).

$(1)_\delta$: *$e^{-1}(C^\varepsilon - D^{\varepsilon/2})$ has a structure of a smooth manifold, $M, \partial_0 M$, and $\partial_1 M$ are $(\delta, 1)$ connected over C^ε, and $(M, \partial_0 M)$ is a (δ, h) cobordism over $C^{2\varepsilon}$ and is a product over D^ε with respect to e.*

(2): *$e = e'$ over $D^\varepsilon \cup \overline{X - C^\varepsilon}$, there is an h cobordism between $\partial_i M$ and $\partial_i M'$, and $(M', \partial_0 M')$ has a product structure over C with respect to e' which is an extension of the structure over D given in $(1)_\delta$.*

Theorem A′. *Suppose X, C, D and ε are as in Theorem* A. *Then there exists a $\delta > 0$, which has the following property* (**).

(**) *For any 1-connected F^5-manifold $(M, \partial_0 M, \partial_1 M)$ and any proper map $e: M \to X$, satisfying the condition $(1')_\delta$ below, M has a mapping cylinder structure over C of diameter less than ε, which is an extension of the one over D given in $(1')_\delta$.*

$(1')_\delta$: *$e^{-1}(C^\varepsilon - D^{\varepsilon/2})$ has a structure of a smooth manifold, $M, \partial_0 M$, and $\partial_1 M$ are $(\delta, 1)$ connected over C^ε, and $(M, \partial_0 M)$ is a (δ, h) cobordism over $C^{2\varepsilon}$ and has a mapping cylinder structure of diameter less than δ over D^ε, i.e., there are a homology 4-manifold N, cell-like maps $f_i : \partial_i M \cap e^{-1}(D^\varepsilon) \to N$, and a homeomorphism $h: e^{-1}(D^\varepsilon) \cong M_{f_0} \bigcup_N M_{f_1}$, which is the identity on $\partial_i M \cap e^{-1}(D^\varepsilon)$.*

Lemma. *Suppose $(W, \partial_0 W, \partial_1 W)$ is a 1-connected F^5-h cobordism. Then there is an* arc *k, such that $(W-k, \partial_0 W - k, \partial_1 W - k)$ has a structure of a smooth proper h cobordism. Moreover, for any $\delta > 0$ and any compact subset D of $[0, \infty)$, there exists a proper map $e: W - k \to [0, \infty)$, such that $W-k$ is a (δ, h) cobordism over D with respect to e.*

Using this lemma, we can show the following corollary, as an application of *Theorems A* and *A′*. (See [3].)

Corollary. *Suppose $(W, \partial_0 W, \partial_1 W)$ is as in Lemma. Then W has a mapping cylinder structure. Moreover, there are an F^5-manifold $(W', \partial_0 W', \partial_1 W')$, which is a product over D with respect to e given in Lemma, and an h cobordism between $(W, \partial_0 W, \partial_1 W)$ and $(W', \partial_0 W', \partial_1 W')$, which is a product outside D.*

The following theorem is also used to prove the main theorem. (Cf. [3], Theorem 4.1.)

Theorem B. *Let X be as in Main Theorem. Then for any compact neighborhood N of N(X), whose boundary components are 3-manifolds, there exist a closed 1-connected F^4-manifold M, a homeomorphism $f: M \cong \overline{X-N} \bigcup_{\partial N = \partial Q} Q$, where Q is some F^4-manifold with boundary, and a homeomorphism $g: X \times \boldsymbol{R} \cong M \times \boldsymbol{R}$, such that*

$$g \mid (\overline{X-N}) \times \boldsymbol{R} \cong (f^{-1} \mid \overline{X-N}) \times \mathrm{id}_{\boldsymbol{R}}.$$

2. Outline of the proof of Main Theorem. We may assume $N(X)$ consists of a single point p. By the condition on X, there is a system $\{N_i\}_{i=0}^{\infty}$ of compact connected neighborhoods of p, such that N_i is contained in int N_{i-1} and is null homotopic in int N_{i-1} and $\bigcap_{i=0}^{\infty} N_i = \{p\}$.

By *Theorem* B, there are a 1-connected closed F^4-manifold

$$M_i \cong \overline{X - N_i} \bigcup_{\partial N_i = \partial Q_i} Q_i$$

and a homeomorphism $f_i : M_i \times \boldsymbol{R} \cong X \times \boldsymbol{R}$ $i=0, 1, \cdots$. For an appropriate sequence of positive numbers $0 < t_0 < t_1 < \cdots, t_n \to \infty$, we may assume $f_i(M_i \times [t_i, \infty))$ is contained in $f_{i-1}(M_{i-1} \times [t_{i-1}, \infty)) \cap X \times (0, \infty)$. Put $W_i = \overline{X \times \boldsymbol{R} - f_{i+1}(M_{i+1} \times (t_{i+1}, \infty))} \cup \overline{f_i(M_i \times (t_i, \infty))}$. Then it is easily seen that $(W_i, \partial_0 W_i, \partial_1 W_i)$ is a 1-connected F^5-h cobordism, which is a product over $\overline{X - N_i} \subset \partial_0 W_i$. By *Corollary*, we obtain a 1-connected F^5-h cobordism $(W'_i, \partial_0 W'_i, \partial_1 W'_i)$, which is a product over $\overline{X - N_{i+1}}$ for an appropriate map $e' : W'_i \to X$. Moreover, there is an h cobordism between $(W_i, \partial_0 W_i, \partial_1 W_i)$ and $(W'_i, \partial_0 W'_i, \partial_1 W'_i)$ which is a product cobordism over $\overline{X - N_i}$. By continuing this process inductively, we can construct homology 4-manifolds $\hat{M}$ and $\hat{X}$, and a cell-like map $f : \hat{X} \to \hat{M}$. Moreover, by the construction $\hat{M}$ (resp. $\hat{X}$) is h cobordant to M_0 (resp. X). Therefore, by *Corollary*, we get a homology 4-manifold N (resp. Z) and cell-like maps $q' : \hat{M} \to N$ and $q'' : M_0 \to N$ (resp. $\hat{q}' : \hat{X} \to Z$ and $\hat{q}'' : X \to Z$). Then after rewriting symbols, we get the required sequence of homology 4-manifolds and cell-like maps between them. The details of the proof will appear elsewhere.

References

[1] J. L. Bryant and J. G. Hollingsworth: Manifold factors that are manifold quotients. Topology, **13**, 19–24 (1974).
[2] J. W. Cannon, J. L. Bryant, and R. C. Lacher: The structure of generalized manifolds having nonmanifold set of trivial dimension. Geometric Topology. Academic Press, pp. 261–300 (1979).
[3] M. Freedman and F. Quinn: A slightly singular 4-manifolds (to appear).
[4] F. Quinn: The stable topology of 4-manifolds (to appear).
[5] ——: Ends of maps I. Ann. Math., **110** (2), 275–331 (1979).
[6] ——: Resolutions of homology manifolds (an informal sketch).