51. *On the Determination of all NB-Structures on BCK-Algebras*

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In this Note, we shall show that all NB-structures on a BCK-algebra are completely determined by a simple way, and the NB-structures give some surprising simplifications of complicated conditions which define various classes of BCK-algebras. Thus the NB-structures on a BCK-algebra may be considered as an auxiliary apparatus.

The NB-structure on a BCK-algebra was independently introduced by the present author and H. Rasiowa (see [1], [3]). To define it, we first recall a definition of BCK-algebras and its basic properties (for detail, see [2]).

A *BCK-algebra* $\langle X; *, 0\rangle$ is an algebra of type $\langle 2, 0\rangle$ satisfying the following conditions (1)–(5).

(1) $((x*y)*(x*z))*(z*y)=0$,

(2) $(x*(x*y))*y=0$,

(3) $x*x=0$,

(4) $0*x=0$,

(5) $x*y=y*x=0$ implies $x=y$.

If we define $x\leq y$ by $x*y=0$, then X is a partially ordered set with respect to $\leq$.

For elements x, y, z in a BCK-algebra;

(6) $x*0=x$,

(7) $(x*y)*z=(x*z)*y$.

If a BCK-algebra X has a greatest element with respect to $\leq$, then X is called to be *bounded*. The greatest element is denoted by 1.

If we define Nx by $1*x$, then the following relations hold:

(8) $N0=1$, $N1=0$,

(9) $Nx*y=Ny*x$ for any x, y.

Generalizing this notion, we arrive at the notion of an NB-algebra.

If a unary operation $\sim$ on a BCK-algebra X satisfies

(10) $\sim x*y=\sim y*x$

for any $x, y\in X$, then X is called an *NB-algebra*.

Let X be an NB-algebra. (10) implies

$$\sim x*0=\sim 0*x.$$

By (6), it follows that

(11) $\sim x=\sim 0*x$.

This shows that $\sim x$ is completely determined by ~ 0 and x. In particular, if $\sim 0 \leq x$, then $\sim x = 0$.

Next, let X be a BCK-algebra. For any fixed element $a \in X$, we define $\sim 0 = a$ and $\sim x = a * x$. Then by (7), we obtain

$$\sim x * y = (a * x) * y = (a * y) * x = \sim y * x.$$

Hence X is an NB-algebra.

From the above consideration, we conclude

Theorem 1. *Any BCK-algebra always has NB-structures. The NB-structures on a BCK-algebra X are completely determined by defining* $\sim 0 = a$, *and* $\sim x = a * x$ *for any fixed element a and any element* $x \in X$.

Theorem 1 and (11) imply

Corollary. *If* $\sim 0 = 1$ *holds in a bounded BCK-algebra X, then the operations N and* $\sim$ *coincide.*

As an application of Theorem 1, we shall give a simple condition that a BCK-algebra is to be positive implicative.

If a BCK-algebra X satisfies

(12) $(x * y) * y = x * y$

for any $x, y \in X$, then it is called to be *positive implicative*.

Let X be a positive implicative BCK-algebra with an NB-structure. Then

$$(\sim x * y) * y = \sim x * y.$$

By (1), (7), we have

$$(\sim x * y) * y = (\sim y * x) * y = (\sim y * y) * x, \qquad \sim x * y = \sim y * x.$$

Hence

(13) $(\sim y * y) * x = \sim y * x.$

Let $x = 0$ in (13). Then, by (6)

$$\sim y * y = \sim y.$$

Conversely, let us suppose that (13) holds in a BCK-algebra X with an NB-structure. Then it is easily seen that X satisfies $(\sim x * y) * y = \sim x * y$. Hence from Theorem 1, we have the following

Theorem 2. *A BCK-algebra X is positive implicative, if and only if for any NB-structure*

$$\sim x * x = \sim x$$

holds.

We shall consider an example.

Example. Let $X = \{0, a, b, c\}$. Let us give the operation $*$ on X by Table I. Then X is a BCK-algebra (see Fig. 1). The NB-structures on X are given by $\sim 0 = 0, a, b, c$. For example, let $\sim 0 = c$. Then $\sim a = c * a = c$, $\sim b = c * b = c$, $c = c * c = 0$. By a similar method, we have Table II.

Consider the type II.

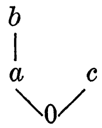

Fig. 1

Table I

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table II

$\sim$	0	a	b	c
I	c	c	c	0
II	b	a	0	b
III	a	c	0	a
IV	0	0	0	0

$$\sim a * a = a * a = 0 \neq \sim a.$$

Hence X is not positive implicative.

Remark. In my Note [1], other three algebras which are B, BN and NBN-algebras are introduced. But we do not find good ways to determine all B, BN, NBN-structures on a BCK-algebras.

By an *NBN-algebra*, we mean a BCK-algebra X with an unary operation $\sim$ satisfying

(14) $\sim x * \sim y \leq y * x$.

This algebra has a close connection with an NB-algebra.

Let X be an NB-algebra. By Theorem 1, for some $a \in X$, $\sim x$ is defined by $a * x$. Hence, from (1), we have

$$\sim x * \sim y = (a * x) * (a * y) \leq y * x.$$

Hence X is an NBN-algebra. Thus we have the following fundamental

Theorem 3. *Any NB-algebra is an NBN-algebra.*

But the converse is not true. To show this, consider the BCK-algebra X in Example. We define $\sim x = a$ for any $x \in X$. Then the operation $\sim$ gives an NBN-structure, which is trivial. This is not interesting, but we can define a non-trivial NBN-structures on X. For example, a non-trivial NBN-structure is defined by

$$\sim 0 = \sim c = a, \qquad \sim a = \sim b = 0$$

as easily verified. This is not an NB-structure on X.

S. Tanaka introduced the notion of a commutative BCK-algebra (for example, see [2]). By a *commutative BCK-algebra X*, we mean a BCK-algebra satisfying

$$x * (x * y) = y * (y * x)$$

for any $x, y \in X$. If we define $x \wedge y = x * (x * y)$, then X is a $\wedge$-semilattice.

Let X be a commutative, NB-algebra. If $\sim \sim x = x$ for some $x \in X$. By Theorem 1 and the commutativity of X, $\sim x = \sim 0 * x$ implies

$$x = \sim \sim x = \sim 0 * (\sim 0 * x) = \sim 0 \wedge x.$$

Hence $x \leq \sim 0$.

Conversely, let $x \leq \sim 0$. Then we have

$$x = \sim 0 \wedge x = \sim \sim x.$$

Hence we have the following

Theorem 4. *In a commutative, NB-algebra, the set of involutions, i.e.* $\sim \sim x = x$ *is given by the set consisting of all elements* x *such that* $x \leq \sim 0$.

References

[1] K. Iséki: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803–807 (1965).

[2] K. Iséki and S. Tanaka: An introduction to the theory of *BCK*-algebras. Math. Japon., **23**, 1–26 (1978).

[3] H. Rasiowa: An algebraic approach to non-classical logics. Studies in Logic and the Foundations of Mathematics. vol. 78, Amsterdam (1974).