35. *Deformation of Linear Ordinary Differential Equations. II*

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In the preceding note [1], we have developed the theory of isomonodromy deformation of linear ordinary differential equations. In particular we defined the τ function for each isomonodromy family, which is a generalization of the theta function in the theory of abelian functions.

In this note we deal with a transformation which changes the exponents of formal monodromy by integer differences (Schlesinger transformation). We also consider the ratio of the transformed τ function to the original one (τ quotient). Finally we shall give elementary examples of τ functions which corresponds to soliton and rational solutions in the theory of inverse scattering.

We use the same notations as [1].

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1. Given an $m\times m$ matrix $Y(x)$ with monodromy property in the sense of [1], we can construct another matrix $Y'(x)$ with the same monodromy data except for integer differences in the exponents of formal monodromy. Schlesinger [2] considered such a transformation in the case of regular singularities. His construction applies equally to the irregular singular case.

Choose integers l_α^ν ($\nu=1,\cdots,n,\infty$; $\alpha=1,\cdots,m$) satisfying the condition (the Fuchs'relation) $\sum_{\nu=1,\cdots,n,\infty}\sum_{\alpha=1}^{m} l_\alpha^\nu=0$, and set $L^{(\nu)}=(\delta_{\alpha\beta}l_\beta^\nu)_{\alpha,\beta=1,\cdots,m}$. A transformation $Y'(x)=R(x)Y(x)$ from $Y(x)$ to $Y'(x)$ is called the Schlesinger transformation of type $\left\{\begin{matrix}\infty & a_1 & \cdots & a_n\\ L^{(\infty)} & L^{(1)} & \cdots & L^{(n)}\end{matrix}\right\}$, if it preserves the monodromy data except for the change of exponents of formal monodromy $T_0^{(\nu)}\mapsto T_0^{(\nu)}+L^{(\nu)}$.

The condition for $R(x)$ so that $Y'(x)$ is the desired matrix is the following.

(1) $$R(x)\hat{Y}^{(\infty)}(x)x^{L^{(\infty)}}=\hat{Y}^{(\infty)\prime}(x),$$

(2) $$R(x)G^{(\nu)}\hat{Y}^{(\nu)}(x)(x-a_\nu)^{-L^{(\nu)}}=G^{(\nu)\prime}\hat{Y}^{(\nu)\prime}(x)$$

with an invertible matrix $G^{(\nu)\prime}$ ($\nu\neq\infty$),

$$(3)\qquad \hat{Y}^{(\nu)\prime}(x)=\begin{cases}\sum_{k=0}^{\infty} Y_k^{(\infty)\prime}x^{-k},\ Y_0^{(\infty)\prime}=1 & (\nu=\infty)\\ \sum_{k=0}^{\infty} Y_k^{(\nu)\prime}(x-a_\nu)^k,\ Y_0^{(\nu)\prime}=1 & (\nu\neq\infty).\end{cases}$$

The multiplier $R(x)$ is uniquely determined from (1)–(3) as a rational function in x with rational coefficients in $a_1,\cdots,a_n$, $Y^{(\nu)}_{k,\alpha\beta}$ ($\nu=1,\cdots,n,\infty$; $k=1,2,\cdots$; $\alpha,\beta=1,\cdots,m$) and $G^{(\nu)}_{\alpha\beta}$ ($\nu=1,\cdots,n$; $\alpha,\beta=1,\cdots,m$).

2. We define the length N of a Schlesinger transformation of type $\begin{Bmatrix}\infty & a_1 & \cdots a_n\\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{Bmatrix}$ by $N=\sum_{\nu=1,\cdots,n,\infty}\sum_{l^\nu_\alpha>0} l^\nu_\alpha$. We say a Schlesinger transformation is elementary if its length is 1. A Schlesinger transformation of length N is decomposed into N elementary transformations. We use the following abbreviated notations for types of elementary transformations: Namely, $\begin{Bmatrix}\nu_0 & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix}$ signifies the type

$$\begin{Bmatrix}\infty & a_1 & \cdots a_n\\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{Bmatrix}\quad\text{with}\quad L^{(\nu)}=-\delta_{\nu\nu_0}E_{\alpha_0}+\delta_{\nu\mu_0}E_{\beta_0}\ (\nu=1,\cdots,n,\infty).$$

Here we set $E_{\alpha_0}=(\delta_{\alpha\alpha_0}\delta_{\beta\alpha_0})_{\alpha,\beta=1,\cdots,m}$.

We shall give the table of multipliers $R(x)$ for elementary transformations.

(4) $\begin{Bmatrix}\infty & \infty\\ \alpha_0 & \beta_0\end{Bmatrix}$: $\quad R(x)=E_{\alpha_0}x+R_0$, $R_{0,\alpha\beta}$ is given by

	$\beta=\alpha_0$	$\beta=\beta_0$	$\beta\neq\alpha_0,\beta_0$
$\alpha=\alpha_0$	$\left(-Y^{(\infty)}_{2,\alpha_0\beta_0}+\sum_{\gamma(\neq\alpha_0)}Y^{(\infty)}_{1,\alpha_0\gamma}Y^{(\infty)}_{1,\gamma\beta_0}\right)\Big/Y^{(\infty)}_{1,\alpha_0\beta_0}$	$-Y^{(\infty)}_{1,\alpha_0\beta_0}$	$-Y^{(\infty)}_{\alpha_0\beta}$
$\alpha=\beta_0$	$1/Y^{(\infty)}_{1,\alpha_0\beta_0}$	0	0
$\alpha\neq\alpha_0,\beta_0$	$-Y^{(\infty)}_{1,\alpha\beta_0}/Y^{(\infty)}_{1,\alpha_0\beta_0}$	0	$\delta_{\alpha\beta}$.

(5) $\begin{Bmatrix}\nu_0 & \nu_0\\ \alpha_0 & \beta_0\end{Bmatrix}$ $(\nu_0\neq\infty)$: $\quad R(x)=1+R_0/(x-a_{\nu_0})$,

$$R_{0,\alpha\beta}=-G^{(\nu_0)}_{\alpha\beta_0}(G^{(\nu_0)-1})_{\alpha_0\beta}/Y^{(\nu_0)}_{1,\alpha_0\beta_0}.$$

(6) $\begin{Bmatrix}\infty & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix}$ $(\mu_0\neq\infty)$: $\quad R(x)=E_{\alpha_0}(x-a_{\mu_0})+R_0$, $R_{0,\alpha\beta}$ is given by

	$\beta=\alpha_0$	$\beta\neq\alpha_0$
$\alpha=\alpha_0$	$\sum_{\gamma(\neq\alpha_0)}Y^{(\infty)}_{1\alpha_0\gamma}G^{(\mu_0)}_{\gamma\beta_0}/G^{(\mu_0)}_{\alpha_0\beta_0}$	$-Y^{(\infty)}_{1,\alpha_0\beta}$
$\alpha\neq\alpha_0$	$-G^{(\mu_0)}_{\alpha\beta_0}/G^{(\mu_0)}_{\alpha_0\beta_0}$	$\delta_{\alpha\beta}$.

(7) $\begin{Bmatrix}\nu_0 & \infty\\ \alpha_0 & \beta_0\end{Bmatrix}$ $(\nu_0\neq\infty)$: $\quad R(x)=1-E_{\beta_0}+R_1/(x-\alpha_{\nu_0})$, $R_{1,\alpha\beta}$ is given by

$\alpha\neq\beta_0$	$-Y^{(\infty)}_{1,\alpha\beta_0}(G^{(\nu_0)-1})_{\alpha_0\beta}/(G^{(\nu_0)-1})_{\alpha_0\beta_0}$
$\alpha=\beta_0$	$(G^{(\nu_0)-1})_{\alpha_0\beta}/(G^{(\nu_0)-1})_{\alpha_0\beta_0}$

(8) $\begin{Bmatrix}\nu_0 & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix}$ $(\nu_0\neq\mu_0,\ \nu_0,\ \mu_0\neq\infty)$: $\quad R(x)=1+R_0/(x-a_{\nu_0})$

$$R_{0,\alpha\beta}=(a_{\nu_0}-a_{\mu_0})G^{(\mu_0)}_{\alpha\beta_0}(G^{(\nu_0)-1})_{\alpha_0\beta}/(G^{(\nu_0)-1}G^{(\mu_0)})_{\alpha_0\beta_0}.$$

3. We define a set of characteristic matrices $G^{(\nu,\mu)(l,k)}$ ($\nu,\mu=1,\cdots,n,\infty$; $l,k\in\boldsymbol{Z}$) as follows. If $l\le 0$ or $k\le 0$ we set

$$G^{(\nu,\mu)(l,k)}=\begin{cases}1 & \text{if } \nu=\mu,\ l+k=1,\ l\le 0\\ -1 & \text{if } \nu=\mu,\ l+k=1,\ k\le 0\\ 0 & \text{otherwise.}\end{cases}$$

In order to define the non trivial part, we prepare the following notations:

$$\left[\sum_{k=0}^{\infty} Y_k x^{-k}\right]_l^{(\infty)}=\sum_{k=0}^{l} Y_k x^{-k} \text{ and}$$

$$\left[\sum_{k=0}^{\infty} Y_k (x-a_\nu)^k\right]_l^{(\nu)}=\sum_{k=0}^{l} Y_k (x-a_\nu)^k\cdot G^{(\nu,\mu)(l,k)}\ (\nu,\mu=1,\cdots,n,\infty;\ l,k\ge 1)$$

are defined by the following identities.

$$(9)\qquad \sum_{k\in\boldsymbol{Z}} G^{(\infty,\infty)(l,k)}x^{1-l-k}=[\hat{Y}^{(\infty)}(x)^{-1}]_{l-1}^{(\infty)}\hat{Y}^{(\infty)}(x)\qquad (l\ge 1),$$

$$\sum_{l\in\boldsymbol{Z}} G^{(\infty,\infty)(l,k)}x^{1-l-k}=-\hat{Y}^{(\infty)}(x)^{-1}[\hat{Y}^{(\infty)}(x)]_{k-1}^{(\infty)}\qquad (k\ge 1).$$

$$(10)\qquad \sum_{k\in\boldsymbol{Z}} G^{(\infty,\nu)(l,k)}x^{1-l}(x-a_\nu)^{k-1}=[\hat{Y}^{(\infty)}(x)^{-1}]_{l-1}^{(\infty)}G^{(\nu)}\hat{Y}^{(\nu)}(x),$$

$$\sum_{l\in\boldsymbol{Z}} G^{(\infty,\nu)(l,k)}x^{-l}(x-a_\nu)^{k}=\hat{Y}^{(\infty)}(x)^{-1}G^{(\nu)}[\hat{Y}^{(\nu)}(x)]_{k-1}^{(\nu)}.$$

$$(11)\qquad \sum_{k\in\boldsymbol{Z}} G^{(\nu,\infty)(l,k)}(x-a_\nu)^{l}x^{-k}=[\hat{Y}^{(\nu)}(x)^{-1}]_{l-1}^{(\nu)}G^{(\nu)-1}\hat{Y}^{(\infty)}(x),$$

$$\sum_{l\in\boldsymbol{Z}} G^{(\nu,\infty)(l,k)}(x-a_\nu)^{l-1}x^{1-k}=\hat{Y}^{(\nu)}(x)^{-1}G^{(\nu)-1}[\hat{Y}^{(\infty)}(x)]_{k-1}^{(\infty)}.$$

$$(12)\qquad \sum_{k\in\boldsymbol{Z}} G^{(\nu,\mu)(l,k)}(x-a_\nu)^{l}(x-a_\mu)^{k-1}=[\hat{Y}^{(\nu)}(x)^{-1}]_{l-1}^{(\nu)}G^{(\nu)-1}G^{(\mu)}\hat{Y}^{(\mu)}(x),$$

$$\sum_{l\in\boldsymbol{Z}} G^{(\nu,\mu)(l,k)}(x-a_\nu)^{l-1}(x-a_\mu)^{k}=-\hat{Y}^{(\nu)}(x)^{-1}G^{(\nu)-1}G^{(\mu)}[\hat{Y}^{(\mu)}(x)]_{k-1}^{(\mu)}.$$

Proposition 1. *We denote by $G^{(\nu,\mu)(l,k)\prime}$ ($\nu,\mu=1,\cdots,n,\infty$; $l,k\in\boldsymbol{Z}$) the characteristic matrices for the transformed matrix $Y'(x)$ by an elementary transformation of type $\begin{Bmatrix}\nu_0 & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix}$. Then for $l,k\ge 1$ we have*

$$(13)\qquad G_{\alpha\beta}^{(\nu,\mu)(l,k)\prime}=\det\begin{pmatrix}G_{\alpha_0\beta_0}^{(\nu_0,\mu_0)(1,1)} & G_{\alpha_0\beta}^{(\nu_0,\mu)(1,k)}\\ G_{\alpha\beta_0}^{(\nu,\mu_0)(l,1)} & G_{\alpha\beta}^{(\nu,\mu)(l,k)}\end{pmatrix}\Big/G_{\alpha_0\beta_0}^{(\nu_0,\mu_0)(1,1)},$$

with the following modifications in the right hand side:

$$l\mapsto\begin{cases}l+1 & \text{if } \nu=\nu_0 \text{ and } \alpha=\alpha_0\\ l-1 & \text{if } \nu=\mu_0 \text{ and } \alpha=\beta_0\end{cases},\qquad k\mapsto\begin{cases}k-1 & \text{if } \mu=\nu_0 \text{ and } \beta=\alpha_0\\ k+1 & \text{if } \mu=\mu_0 \text{ and } \beta=\beta_0\end{cases}.$$

4. We denote by $q\begin{Bmatrix}\infty & a_1 & \cdots & a_n\\ L^{(\infty)} & L^{(1)} & \cdots & L^{(n)}\end{Bmatrix};Y(x)\Big\}$ the ratio of the τ function for the transformed matrix $Y'(x)$ by a Schlesinger transformation of type $\begin{Bmatrix}\infty & a_1 & \cdots & a_n\\ L^{(\infty)} & L^{(1)} & \cdots & L^{(n)}\end{Bmatrix}$ to the τ function for the original matrix $Y(x)$.

Proposition 2. *For an elementary transformation of type $\begin{Bmatrix}\nu_0 & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix}$ the τ quotient $q\begin{Bmatrix}\nu_0 & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix};Y(x)\Big\}$ is given by*

$$(14)\qquad q\begin{Bmatrix}\nu_0 & \mu_0\\ \alpha_0 & \beta_0\end{Bmatrix};Y(x)\Big\}=G_{\alpha_0\beta_0}^{(\nu_0,\mu_0)(1,1)}=\begin{cases}Y_{1,\alpha_0\beta_0}^{(\nu_0)} & \text{if } \nu_0=\mu_0\\ G_{\alpha_0\beta_0}^{(\mu_0)} & \text{if } \nu_0=\infty,\ \mu_0\ne\infty\\ (G^{(\nu_0)-1})_{\alpha_0\beta_0} & \text{if } \nu_0\ne\infty,\ \mu_0=\infty\\ (G^{(\nu_0)-1}G^{(\mu_0)})/(a_{\mu_0}-a_{\nu_0}) & \\ & \text{if } \nu_0,\mu_0\ne\infty,\ \nu_0\ne\mu_0.\end{cases}$$

In general, we have the following

Theorem 3. *Let* $W\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}; Y(x)\right\}$ *be the following* $N\times N$ *(*N*: the length) matrix,*

$$W\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}; Y(x)\right\}=(G_{\alpha\beta}^{(\nu,\mu)(l,k)})_{\substack{\nu,\mu=1,\cdots,n,\infty;\alpha,\beta=1,\cdots,m\\ l=1,\cdots,N_\alpha^{-\nu};k=1,\cdots,N_\beta^{+\mu}}}, \tag{15}$$

where $N_\alpha^{+\nu}=\max(l_\alpha^\nu,0)$ *and* $N_\alpha^{-\nu}=-\min(l_\alpha^\nu,0)$. *Then we have*

$$q\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}; Y(x)\right\}=\det W\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}; Y(x)\right\}. \tag{16}$$

Moreover, the characteristic matrices $G^{(\nu,\mu)(l,k)\prime}$ $(\nu,\mu=1,\cdots,n,\infty;l,k=1,2,\cdots)$ *for the transformed matrix* $Y'(x)$ *by the Schlesinger transformation of type* $\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}\right\}$ *is given by*

$$q\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}\right\} G_{\alpha_0\beta_0}^{(\nu_0,\mu_0)(l_0,k_0)\prime} \tag{17}$$

$$=\det\begin{pmatrix} W\left\{\begin{matrix}\infty & a_1 & \cdots a_n \\ L^{(\infty)} & L^{(1)} & \cdots L^{(n)}\end{matrix}\right\} & (G_{\alpha\beta_0}^{(\nu,\mu_0)(l,k_0+l_{\beta_0}^{\mu_0})})_{\substack{\nu=1,\cdots,n,\infty\\ \alpha=1,\cdots,m\\ l=1,\cdots,l_\alpha^{-\nu}}} \\ (G_{\alpha_0\beta}^{(\nu_0,\mu)(l_0-l_{\alpha_0}^{\nu_0},k)})_{\substack{\mu=1,\cdots,n,\infty\\ \beta=1,\cdots,m\\ k=1,\cdots,l_\beta^{+\mu}}} & G_{\alpha_0\beta_0}^{(\nu_0,\mu_0)(l_0-l_{\alpha_0}^{\nu_0},k_0+l_{\beta_0}^{\mu_0})} \end{pmatrix} \quad (l_0,k_0\geq 1).$$

Remark 1. We say that $x=a$ is regular for $Y(x)$ if $Y(x)$ is holomorphic and invertible at $x=a$. In this case we can choose an $m\times m$ invertible constant matrix C and consider the point $x=a$ as a regular singular point with the connection matrix C and the exponents $(0,\cdots,0)$ of formal monodromy. Then (14) implies that $Y(a)$ is also expressible as a τ quotient.

5. *Soliton solution* (cf. [3], [4]). Take $Y(x)=e^{T(x)}$ where $T(x)$ is a polynomial in x with $m\times m$ diagonal matrices as coefficients such that $T(0)=0$. We choose an integer N, Nm points $a_1,\cdots,a_{Nm}$ and Nm $m\times m$ matrices $C_1,\cdots,C_{Nm}$ which are supposed to be the connection matrices at $a_1,\cdots,a_{Nm}$, respectively. The Schlesinger transformation of type $\left\{\begin{matrix}\infty & a_1 \cdots a_{Nm} \\ NI & E_1\cdots E_1\end{matrix}\right\}$ for $Y(x)$ is given by a multiplier $R(x)$ of the form $R(x)=x^N+Y_1x^{N-1}+\cdots+Y_N$. Theorem 3 reads as

$$q\left\{\begin{matrix}\infty & a_1 \cdots a_{Nm} \\ NI & E_1\cdots E_1\end{matrix}; Y(x)\right\}=\det W, \tag{18}$$

$$(Y_1,\cdots,Y_N)=-W^{(N)}W^{-1}, \tag{19}$$

where

$$W=\begin{bmatrix} W^{(N-1)} \\ \vdots \\ W^{(0)} \end{bmatrix} \quad \text{and} \quad W^{(l)}=\left(a_1^l e^{T(a_1)}C_1^{-1}\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix},\cdots,a_{Nm}^l e^{T(a_{Nm})}C_{Nm}^{-1}\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right).$$

6. *Rational solution* (cf. [5]). Choose integers $\lambda_1,\cdots,\lambda_m$ and $\mu_1,\cdots,\mu_m$ such that $\lambda_1\geq\cdots\geq\lambda_m$, $\mu_1\geq\cdots\geq\mu_m$, $\lambda_1+\cdots+\lambda_m=\mu_1+\cdots+\mu_m$

and $\mu_1+\cdots+\mu_\alpha\leq\lambda_1+\cdots+\lambda_\alpha$ $(\alpha=1,\cdots,m-1)$. We also choose an $m\times m$ matrix C as the connection matrix at $x=0$. Take $Y(x)=x^{\lambda m}e^{T(x)}$ and consider the Schlesinger transformation of type

$$\begin{Bmatrix} \infty & 0 \\ L^{(\infty)}=(\delta_{\alpha\beta}(\lambda_\beta-\mu_\beta)) & L^{(0)}=(\delta_{\alpha\beta}(\lambda_\beta-\lambda_m)) \end{Bmatrix} \quad \text{for } Y(x).$$

The multiplier $R(x)$ is of the form

$$R(x)=1+Y_1x^{-1}+\cdots+(Y_{\mu_1-\lambda_m}x^{\lambda_m-\mu_1})x^{\begin{pmatrix}\mu_1-\lambda_m & & \\ & \ddots & \\ & & \mu_m-\lambda_m\end{pmatrix}}.$$

We define a sequence of integers α_l by $\alpha_l=\alpha$ if $\mu_\alpha-\lambda_\alpha+1\leq l\leq\mu_{\alpha-1}-\lambda_\alpha$. Then for $\alpha_l+1\leq\alpha\leq m$ α-th column of Y_l is zero. We denote by $\tilde{Y}_l$ the $m\times\alpha_l$ non zero part of Y_l. We also define a row vector $\boldsymbol{c}_{\alpha\beta}^{(l)}$ $(l\geq 0,$ $1\leq\alpha,\ \beta\leq m)$ of the size $\lambda_\alpha-\lambda_\beta$ by

$$\boldsymbol{c}_{\alpha\beta}^{(l)}=\left((C^{-1})_{\alpha\beta},\cdots,\left(\frac{1}{(\lambda_k-\lambda_m-1)!}\frac{d^{\lambda_k-\lambda_m-1}}{dx^{\lambda_k-\lambda_m-1}}e^{T(x)}\bigg|_{x=0}\right)_{\alpha\alpha}(C^{-1})_{\alpha\beta}\right)Q_\beta^l$$

where $Q=\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{pmatrix}$ (of size $\lambda_\alpha-\lambda_m+1$), and an $m\times(\lambda_\alpha-\lambda_m)$ matrix $W_\alpha^{(l)}$ $(l\geq 0,\ 1\leq\alpha\leq m-1)$ by $W_\alpha^{(l)}={}^t({}^t\boldsymbol{c}_{1\alpha}^{(l)},{}^t\boldsymbol{c}_{2\alpha}^{(\mu_2-\mu_1+l)},\cdots,\boldsymbol{c}_{m\alpha}^{(\mu_m-\mu_1+l)})$. We denote by $W_{\alpha,\alpha'}^{(l)}$ the $\alpha'\times(\lambda_\alpha-\lambda_m)$ matrix made of the first α' rows of $W_\alpha^{(l)}$. Then Theorem 3 reads as

$$\begin{Bmatrix}\infty & 0 \\ L^{(\infty)} & L^{(0)}\end{Bmatrix}=\det W, \tag{20}$$

$$(\tilde{Y}_1,\cdots,\tilde{Y}_N)=-(W_1^{(N)},\cdots,W_{m-1}^{(N)})W^{-1}, \tag{21}$$

where $W=(W_{\alpha,\alpha_j}^{(N-j)})_{j=1,\cdots,N;\ k=1,\cdots,m-1}$.

Remark 2. In both examples, the τ function for the original matrix $Y(x)$ is 1. Hence the formula (18) or (20) gives the τ function for the transformed matrix $Y'(x)=R(x)Y(x)$.

References

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