34. Deformation of Linear Ordinary Differential Equations. I

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In this article we report on the general theory of isomonodromic deformations for a system of linear ordinary differential equations (1), having irregular singularities of arbitrary rank. A general scheme for such deformations was constructed by L. Schlesinger [1] for equations with regular singularities, and was recently extended by K. Ueno [2], B. Klares [10] to the case admitting irregular singularities. The main results of the present note are (i) proof of complete integrability of the nonlinear deformation equations (§§ 2–3), and (ii) introduction of the notion of τ function (§ 4).

Details of this and the forthcoming note II will be published elsewhere.

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1. Let $a_1, \cdots, a_n, a_\infty = \infty$ be distinct points on $\boldsymbol{P}^1$. We consider a system of linear ordinary differential equations with rational coefficients

$$(1) \qquad \frac{dY}{dx} = A(x)Y, \qquad A(x) = \sum_{\nu=1}^{n} \sum_{k=0}^{r_\nu} \frac{A_{\nu,-k}}{(x-a_\nu)^{k+1}} - \sum_{k=1}^{r_\infty} A_{\infty,-k} x^{k-1}$$

where $A_{\nu,-k}$ are $m \times m$ constant matrices. We set $A_{\infty 0} = -\sum_{\nu=1}^{n} A_{\nu 0}$. The leading coefficients $A_{\nu,-r_\nu}$ at $x=a_\nu$ are assumed to be diagonalized as

$$(2) \qquad A_{\nu,-r_\nu} = G^{(\nu)} T^{(\nu)}_{-r_\nu} G^{(\nu)-1} \qquad (\nu = 1, \cdots, n, \infty)$$

$T^{(\nu)}_{-r_\nu}$: diagonal with eigenvalues mutually distinct (if $r_\nu \geq 1$) or distinct modulo integers (if $r_\nu = 0$).

In the sequel we assume $A_{\nu,-r_\nu} = T^{(\nu)}_{-r_\nu}$ and choose $G^{(\infty)} = 1$. Along with (1) we consider equivalent systems with diagonalized leading term at $x = a_\nu$

$$(3) \qquad \frac{dY^{(\nu)}}{dx} = A^{(\nu)}(x) Y^{(\nu)}, \qquad A^{(\nu)}(x) = G^{(\nu)-1} A(x) G^{(\nu)}.$$

Equation (1) is specified by the following data

$$(4) \qquad a_\nu, A_{\nu 0}, \cdots, A_{\nu,-r_\nu+1}, T^{(\nu)}_{-r_\nu}, G^{(\nu)} \qquad (\nu = 1, \cdots, n);\ A_{\infty -1}, \cdots, A_{\infty - r_\infty}.$$

We denote by $\mathfrak{M}$ the affine manifold of "singularity data" (4).

Equation (3) has a unique formal solution of the following form ([5][6]):

(5) $$Y^{(\nu)}(x)=\hat{Y}^{(\nu)}(x)e^{T^{(\nu)}(x)},\quad \hat{Y}^{(\nu)}(x)=\sum_{k=0}^{\infty}Y_k^{(\nu)}z_\nu^k,\quad Y_0^{(\nu)}=1$$
$$T^{(\nu)}(x)=\sum_{k=1}^{r_\nu}T_{-k}^{(\nu)}\frac{z_\nu^{-k}}{(-k)}+T_0^{(\nu)}\log z_\nu:\ \text{diagonal}$$

where

(6) $$z_\nu=x-a_\nu\ (\nu\neq\infty),\qquad =\frac{1}{x}\ (\nu=\infty).$$

We call $T_0^{(\nu)}$ the exponent of formal monodromy at $x=a_\nu$. The coefficients $Y_k^{(\nu)}$, $T_k^{(\nu)}$ are determined from $d\hat{Y}^{(\nu)}/dz_\nu+\hat{Y}^{(\nu)}dT^{(\nu)}/dz_\nu=A^{(\nu)}(x)dx/dz_\nu\hat{Y}^{(\nu)}$. Let

$$A^{(\nu)}(x)=\sum_{k=-r_\nu}^{\infty}A_k^{(\nu)}(x-a_\nu)^{k-1}\ (\nu\neq\infty),\qquad =-\sum_{k=-r_\infty}^{\infty}A_k^{(\infty)}\left(\frac{1}{x}\right)^{k+1}\ (\nu=\infty)$$

be the Laurent expansion at $x=a_\nu$. We set $\hat{Y}^{(\nu)}(x)=F^{(\nu)}(z_\nu)D^{(\nu)}(z_\nu)$, $F^{(\nu)}(z_\nu)=\sum_{k=0}^{\infty}F_k^{(\nu)}z_\nu^k$, $D^{(\nu)}(z_\nu)=\sum_{k=0}^{\infty}D_k^{(\nu)}z_\nu^k$, $F_0^{(\nu)}=1$, $D_0^{(\nu)}=1$, and $F_k^{(\nu)}$ diagonal free, $D_k^{(\nu)}$ diagonal for $k\geq 1$; then the following recursion relations hold (with $A_{-r_\nu}^{(\nu)}=T_{-r_\nu}^{(\nu)}$).

(7) $$[F_k^{(\nu)},T_{-r_\nu}^{(\nu)}]=\sum_{j=1}^{k}(A_{-r_\nu+j}^{(\nu)}F_{k-j}^{(\nu)}-F_{k-j}^{(\nu)}T_{-r_\nu+j}^{(\nu)})-I_{k-r_\nu}^{(\nu)}$$
$$Y_k^{(\nu)}=\sum_{j=0}^{k}F_{k-j}^{(\nu)}D_j^{(\nu)}.$$

Here we have set $T_k^{(\nu)}=0$ $(k\geq 1)$ and

(8) $$I_k^{(\nu)}=0\ (k\leq 0),\qquad =kD_k^{(\nu)}-\sum_{j=1}^{k-1}I_{k-j}^{(\nu)}D_j^{(\nu)}\ (k\geq 1).$$

For instance

$$[F_1^{(\nu)},T_{-r_\nu}^{(\nu)}]=A_{-r_\nu+1}^{(\nu)}-T_{-r_\nu+1}^{(\nu)},$$
$$[F_2^{(\nu)},T_{-r_\nu}^{(\nu)}]=A_{-r_\nu+1}^{(\nu)}F_1^{(\nu)}+A_{-r_\nu+2}^{(\nu)}-F_1^{(\nu)}T_{-r_\nu+1}^{(\nu)}-T_{-r_\nu+2}^{(\nu)},\ \cdots,$$
$$D_1^{(\nu)}=\left(\sum_{k=1}^{r_\nu+1}A_{-r_\nu+k}^{(\nu)}F_{r_\nu+1-k}^{(\nu)}\right)_D,$$
$$2D_2^{(\nu)}-D_1^{(\nu)2}=\left(\sum_{k=1}^{r_\nu+2}A_{-r_\nu+k}^{(\nu)}F_{r_\nu+2-k}^{(\nu)}\right)_D,\ \cdots$$

where $(X)_D=(\delta_{\alpha\beta}X_{\alpha\alpha})$ denotes the diagonal part of a matrix $X=(X_{\alpha\beta})$. Quantities $Y_k^{(\nu)}$, $T_k^{(\nu)}$, $F_k^{(\nu)}$ and $D_k^{(\nu)}$ are rational functions on $\mathcal{M}$.

Next choose a set of sectors $\mathcal{S}_1^{(\nu)},\cdots,\mathcal{S}_{k_\nu+1}^{(\nu)}$ in the universal covering manifold $\tilde{\mathcal{R}}$ of $\mathcal{R}=\boldsymbol{P}^1-\{a_1,\cdots,a_n,\infty\}$, such that $\mathcal{S}_l^{(\nu)}\cap\mathcal{S}_{l+1}^{(\nu)}\neq\phi$ and $\bigcup_{l=1}^{k_\nu+1}\pi(\mathcal{S}_l^{(\nu)})=V_\nu-\{a_\nu\}$. Here $\pi:\tilde{\mathcal{R}}\to\mathcal{R}$ is a projection and V_ν is a small neighborhood of a_ν. For example we choose

(9) $$\pi(\mathcal{S}_{l,\delta}^{(\nu)})=\left\{x\in V_\nu\,\middle|\,\frac{\pi(l-1)}{r_\nu}-\delta<\arg z_\nu<\frac{\pi l}{r_\nu}\right\},\qquad k_\nu=2r_\nu$$

with $\delta>0$ small. For $r_\nu=0$ set $\mathcal{S}_1^{(\nu)}=V_\nu-\{a_\nu\}$.

It is known that there exists a unique actual solution $Y_l^{(\nu)}(x)$ of (3), holomorphic and invertible in $\mathcal{S}_{l,\delta}^{(\nu)}$, having the asymptotic expansion

(10) $$Y_l^{(\nu)}(x)\sim\hat{Y}^{(\nu)}(x)e^{T^{(\nu)}(x)}\qquad\text{in }\mathcal{S}_{l,\delta}^{(\nu)}.$$

The Stokes multipliers $S_l^{(\nu)}$ are defined to be the constant matrices satisfying

(11) $$Y_{l+1}^{(\nu)}(x)=Y_l^{(\nu)}(x)S_l^{(\nu)}, \qquad l=1,\cdots,k_\nu.$$

Since $Y_1^{(\infty)}(x)$ and $G^{(\nu)}Y_1^{(\nu)}(x)$ satisfy the same equation (1), the connection matrices $C^{(\nu)}$ are defined through

(12) $$Y_1^{(\infty)}(x)=G^{(\nu)}Y_1^{(\nu)}(x)C^{(\nu)}, \qquad \nu=1,\cdots,n,\infty \ (C^{(\infty)}=1).$$

The monodromy matrix for $Y_1^{(\infty)}(x)$ at a_ν is given by

(13) $$Y_1^{(\infty)}(x)|_{z_\nu\mapsto e^{2\pi i}z_\nu}=Y_1^{(\infty)}(x)M_\nu,$$
$$M_\nu=C^{(\nu)-1}e^{2\pi i T_0^{(\nu)}}S_{k_\nu}^{(\nu)-1}\cdots S_1^{(\nu)-1}C^{(\nu)}.$$

2. Now we are concerned with a family $\{Y(x,t)\}$ of functions parametrized by some $t\in \boldsymbol{C}^N$, such that the monodromy property is preserved under the variation of parameters in the following sense:

(14) The Stokes multipliers $S_l^{(\nu)}$, connection matrices $C^{(\nu)}$ and the exponents of formal monodromy $T_0^{(\nu)}$ are independent of t.

In particular the monodromy matrices M_ν are also invariant by (13). Following the scheme developed by Schlesinger [1] and Ueno [2], it is shown that a necessary and sufficient condition for (14) is expressed in terms of a system of total differential equations with respect to t; for $Y_l^{(\nu)}(x)$ it is linear

(15) $$dY_l^{(\nu)}=\Omega^{(\nu)}Y_l^{(\nu)} \qquad (\nu=1,\cdots,n,\infty),$$

and for the coefficient matrices it is non-linear

(16) $$dA^{(\nu)}=\frac{\partial\Omega^{(\nu)}}{\partial x}+[\Omega^{(\nu)},A^{(\nu)}]$$
$$dG^{(\nu)}=\Theta^{(\nu)}G^{(\nu)}.$$

Here d denotes the exterior differentiation with respect to the parameters t, and $\Omega^{(\nu)}=\Omega^{(\nu)}(x)$, $\Theta^{(\nu)}$ are matrices of 1-forms calculated from $A(x)$, $G^{(\nu)}$ as follows. First we define matrices $\Phi_k^{(\nu)}$ of 1-forms through

(17) $$\hat{Y}^{(\nu)}(x)d'T^{(\nu)}(x)\cdot\hat{Y}^{(\nu)}(x)^{-1}=\sum_{k=-r_\nu-1}^{\infty}\Phi_k^{(\nu)}z_\nu^k$$

with

(18) $$d'T^{(\nu)}(x)=\sum_{k=1}^{r_\nu}dT_{-k}^{(\nu)}\frac{z_\nu^{-k}}{(-k)}-\sum_{k=0}^{r_\nu}T_{-k}^{(\nu)}z_\nu^{-k-1}da_\nu.$$

For $\nu=\infty$, $\Phi_{-r_\infty-1}^{(\infty)}=0$ and the second term of (18) is absent. Explicitly we have ($T_{-k}^{(0)}=0$ if $k>r_\nu$)

(19) $$\Phi_{-k}^{(\nu)}=\frac{dT_{-k}^{(\nu)}}{(-k)}-T_{-k+1}^{(\nu)}da_\nu$$
$$+\sum_{\substack{k_1+\cdots+k_s=k'-k\\ k_1,\cdots,k_s,k'\geq 1}}\left[\frac{dT_{-k'}^{(\nu)}}{(-k')}-T_{-k'+1}^{(\nu)}da_\nu,(-Y_{k_1}^{(\nu)})\right](-Y_{k_2}^{(\nu)})\cdots(-Y_{k_s}^{(\nu)})$$
$$(\nu=1,\cdots,n,\ 1\leq k\leq r_\nu+1)$$

(20) $$\Phi_{-k}^{(\infty)}=\frac{dT_{-k}^{(\infty)}}{(-k)}+\sum_{\substack{k_1+\cdots+k_s=k'-k\\ k_1,\cdots,k_s,k'\geq 1}}\left[\frac{dT_{-k'}^{(\infty)}}{(-k')},(-Y_{k_1}^{(\infty)})\right](-Y_{k_2}^{(\infty)})\cdots(-Y_{k_s}^{(\infty)})$$
$$(0\leq k\leq r_\infty).$$

In (20) $dT_{-k}^{(\infty)}/(-k)$ is omitted if $k=0$. In terms of $\Phi_k^{(\nu)}$, $\Omega^{(\nu)}$ and $\Theta^{(\nu)}$

are given by

$$\Omega^{(\infty)}(x)=\sum_{\nu=1}^{n}\sum_{k=1}^{r_\nu+1} G^{(\nu)}\Phi^{(\nu)}_{-k}G^{(\nu)-1}\frac{1}{(x-a_\nu)^k}+\sum_{k=0}^{r_\infty}\Phi^{(\infty)}_{-k}x^k \tag{21}$$

$$\Omega^{(\nu)}(x)=G^{(\nu)-1}(\Omega^{(\infty)}(x)-\Theta^{(\nu)})G^{(\nu)} \qquad (\nu\neq\infty) \tag{22}$$

$$\Theta^{(\nu)}=G^{(\nu)}(-\Phi^{(\nu)}_0+Y^{(\nu)}_1 da_\nu)G^{(\nu)-1}+\sum_{k=0}^{r_\infty}\Phi^{(\infty)}_{-k}a^k_\nu + \sum_{\mu(\neq\nu)}\sum_{k=1}^{r_\mu+1} G^{(\mu)}\Phi^{(\mu)}_{-k}G^{(\mu)-1}\frac{1}{(a_\nu-a_\mu)^k}. \tag{23}$$

The system (16) together with $d\Omega^{(\nu)}=\Omega^{(\nu)2}$ guarantees the integrability of (1)+(15). It is shown (cf. (29)) that $d\Omega^{(\nu)}=\Omega^{(\nu)2}$ is a consequence of (16), and that there is a canonical choice of deformation parameters t (see (27)′ below).

3. Actually (16) contains redundancy and is reducible to a smaller set of equations. We set

$$\Xi^{(\nu)}(x)=dA^{(\nu)}(x)-\frac{\partial\Omega^{(\nu)}}{dx}-[\Omega^{(\nu)}(x), A^{(\nu)}(x)]. \tag{24}$$

It is shown that the Laurent expansion of (24) at a_ν has the form

$$\Xi^{(\infty)}(x)=\sum_{k=-r_\infty+1}^{\infty}\Xi^{(\infty)}_k\left(\frac{1}{x}\right)^{k+1} \tag{25}$$

$$\Xi^{(\nu)}(x)=\sum_{k=-r_\nu+1}^{\infty}\Xi^{(\nu)}_k(x-a_\nu)^{k-1} \qquad (\nu\neq\infty).$$

Let $\mathcal{J}$ denote the ideal of differential forms on $\mathfrak{N}$ generated by the following 1-forms

$$(\Xi^{(\nu)}_{-r_\nu+1})_{\alpha\beta},\cdots,(\Xi^{(\nu)}_{-1})_{\alpha\beta} \qquad (\nu=1,\cdots,n,\infty;\alpha\neq\beta) \tag{26}$$

$$(\Xi^{(\nu)}_0)_{\alpha\beta},(dG^{(\nu)}-\Theta^{(\nu)}G^{(\nu)})_{\alpha\beta} \qquad (\nu=1,\cdots,n;\text{all }\alpha,\beta).$$

It is easy to see that the totality of following quantities (27)′+(27)″ (regarded as rational functions on $\mathfrak{N}$) constitute a coordinate system on $\mathfrak{N}$:

(27)′ $\quad a_1,\cdots,a_n;\ t^{(\nu)}_{-k\alpha} \qquad (1\leq k\leq r_\nu, 1\leq\alpha\leq m, \nu=1,\cdots,n,\infty)$

(27)″ $\quad (A^{(\nu)}_{-r_\nu+1})_{\alpha\beta},\cdots,(A^{(\nu)}_{-1})_{\alpha\beta} \qquad (\nu=1,\cdots,n,\infty;\alpha\neq\beta)$

$\qquad (A^{(\nu)}_0)_{\alpha\beta},(G^{(\nu)})_{\alpha\beta} \qquad (\nu=1,\cdots,n;\text{all }\alpha,\beta).$

Here $t^{(\nu)}_{-k\alpha}$ are the entries of the diagonal matrix $T^{(\nu)}_{-k}$ appearing in the exponential (5). Note that the 1-forms (26) are of the form $dy_i-\Sigma f_{ij}(x,y)dx_j$ where x_j (resp. y_i) signifies the coordinate (27)′ (resp. (27)″).

The following hold.

Theorem 1. *All the coefficients of the rational function* (24) *belong to the ideal* $\mathcal{J}$.

Theorem 2. *The ideal* $\mathcal{J}$ *is closed*: $d\mathcal{J}\subset\mathcal{J}$, *where* d *denotes the exterior differentiation on the manifold* $\mathfrak{N}$.

By Frobenius' theorem the system $\mathcal{J}=0$ is then completely integrable. In conclusion, the deformation equations (16) are equivalent to a completely integrable Pfaffian system ((26) set equal to 0), whose independent variables and the unknown functions are chosen to be (27)′ and (27)″, respectively.

In the course of the proof of Theorems 1, 2, we find that the following also belog to $\mathcal{J}$.

(28) $$d\hat{Y}^{(\nu)}+\hat{Y}^{(\nu)}d'T^{(\nu)}-\Omega^{(\nu)}\hat{Y}^{(\nu)} \qquad (\nu=1,\cdots,n,\infty)$$

(29) $$d\Omega^{(\nu)}-\Omega^{(\nu)2} \quad (\nu=1,\cdots,n,\infty), \qquad d\Theta^{(\nu)}-\Theta^{(\nu)2} \quad (\nu=1,\cdots,n)$$

(30) $$dT_0^{(\nu)} \qquad (\nu=1,\cdots,n,\infty).$$

In particular (28) gives an infinite system of nonlinear differential equations among $Y_k^{(\nu)}$'s. For instance the differential $d_\infty\hat{Y}^{(\infty)}$ involving $d'T^{(\infty)}$ is given by

(31) $$d_\infty Y_k^{(\infty)}=-\sum_{\substack{k_1+\cdots+k_s=k'+k\\ k_1,\cdots,k_s,k'\geq 1,k_s\geq k}}\left[\frac{dT_{-k'}^{(\infty)}}{-k'},(-Y_{k_1}^{(\infty)})\right](-Y_{k_2}^{(\infty)})\cdots(-Y_{k_s}^{(\infty)}) \qquad (k=1,2,\cdots).$$

By construction the singularities of (the coefficients of) the deformation equations are confined to

(32) $$a_\mu=a_\nu \quad \text{for some } \mu\neq\nu,$$
$$t_{-r_\nu\alpha}^{(\nu)}=t_{-r_\nu\beta}^{(\nu)} \quad \text{for some } \alpha\neq\beta, \quad \nu \ \text{with} \ r_\nu\geq 1.$$

We expect that the deformation equations are "of Painlevé type", namely:

Conjecture 1. *Aside from the fixed critical varieties* (32), *a general solution to* (16) *can have at most poles.*

4. To each solution of the deformation equations, there is canonically associated a closed 1-form. We set

(33) $$\omega=-\sum_{\nu=1,\cdots,n,\infty}\operatorname*{Res}_{x=a_\nu}\operatorname{trace}\hat{Y}^{(\nu)}(x)^{-1}\frac{\partial\hat{Y}^{(\nu)}}{\partial x}d'T^{(\nu)}(x)dx.$$

Here the residue of a formal Laurent series $\sum_{k=-r}^{\infty}f_k(x-a_\nu)^k dx$ $\left(\text{resp. } \sum_{k=-r}^{\infty}f_k x^{-k}dx\right)$ is defined to be f_{-1} (resp. $-f_1$). In terms of $Y_k^{(\nu)}$ (33) reads

(34) $$\omega=\omega_1+\cdots+\omega_n+\omega_\infty,$$
$$\omega_\nu=\operatorname{trace}\left(\sum_{k=1}^{r_\nu}Z_k^{(\nu)}dT_{-k}^{(\nu)}+\sum_{k=1}^{r_\nu+1}kZ_k^{(\nu)}T_{-k+1}^{(\nu)}da_\nu\right)$$
(for $\nu=\infty$ the second term is omitted)

where $Z_1^{(\nu)}=Y_1^{(\nu)}$, $Z_2^{(\nu)}=Y_2^{(\nu)}-\frac{1}{2}Y_1^{(\nu)2}$, $Z_3^{(\nu)}=Y_3^{(\nu)}-\frac{2}{3}Y_1^{(\nu)}Y_2^{(\nu)}-\frac{1}{3}Y_2^{(\nu)}Y_1^{(\nu)}+\frac{1}{3}Y_1^{(\nu)3},\cdots,$

(35) $$Z_k^{(\nu)}=-\frac{1}{k}\sum_{s=1}^{k}\sum_{\substack{k_1+\cdots+k_s=k\\ k_1,\cdots,k_s\geq 1}}k_s(-Y_{k_1}^{(\nu)})\cdots(-Y_{k_s}^{(\nu)}).$$

Theorem 3. $d\omega\in\mathcal{J}$.

Hence to each solution of (16) there exists a function τ, unique up to a multiplicative constant, such that (cf. [7] [8])

(36) $$\omega=d\log\tau.$$

We conjecture that this τ function is free form movable poles.

Conjecture 2. *The τ function is holomorphic everywhere on the universal covering manifold of $C^N - \{$the fixed critical varieties (32)$\}$.* Here $N=n+m \sum_{\nu=1,\cdots,n,\infty} r_\nu$ denotes the number of independent variables (27)$'$.

Conjectures 1, 2 are known to be true in the case where (16) reduces to the Painlevé equations (cf. [9]).

Remark (cf. [10]). If we fix $T^{(\nu)}_{-k}$'s and vary $a_1, \cdots, a_n$ only, then the deformation equations (16) and the 1-form ω (33) reduce respectively to

$$(16)' \quad dA_{\nu,-k} = \sum_{\mu(\neq\nu)} \sum_{l=0}^{r_\mu} \sum_{m=0}^{r_\nu - k} (-)^l \binom{l+m}{m} [A_{\mu,-l}, A_{\nu,-m-k}] \frac{d(a_\mu - a_\nu)}{(a_\mu - a_\nu)^{l+m+1}} - \sum_{l=1}^{r_\infty} \sum_{m=0}^{\min(l-1, r_\nu - k)} \binom{l-1}{m} [A_{\infty,-l}, A_{\nu,-m-k}] a_\nu^{l-m-1} da_\nu$$

$$dA_{\infty,-k} = -\sum_{\nu=1}^{n} \sum_{l=1}^{r_\infty - k} \sum_{m=0}^{\min(l-1, r_\nu)} \binom{l-1}{m} [A_{\nu,-m}, A_{\infty,-k-l}] a_\nu^{l-m-1} da_\nu$$

$$dG^{(\nu)} \cdot G^{(\nu)-1} = \sum_{\mu(\neq\nu)} \sum_{k=0}^{r_\mu} A_{\mu,-k} \frac{d(a_\nu - a_\mu)}{(a_\nu - a_\mu)^{k+1}} - \sum_{k=1}^{r_\infty} A_{\infty,-k} a_\nu^{k-1} da_\nu$$

$$(33)' \quad \omega = \frac{1}{2} \sum_{\mu\neq\nu} \sum_{k=0}^{r_\mu} \sum_{l=0}^{r_\nu} (-)^k \binom{k+l}{k} \operatorname{trace} A_{\mu,-k} A_{\nu,-l} \frac{d(a_\mu - a_\nu)}{(a_\mu - a_\nu)^{k+l+1}} - \sum_{\nu=1}^{n} \sum_{k=0}^{r_\nu} \sum_{l=k+1}^{r_\infty} \binom{l-1}{k} \operatorname{trace} A_{\nu,-k} A_{\infty,-l} a_\nu^{-k+l-1} da_\nu.$$

References

[1] L. Schlesinger: J. Reine u. Angew. Math., **141**, 96 (1912).

[2] K. Ueno: Master's Thesis. Kyoto Univ. (1979) (in Japanese).
——: Monodromy preserving deformation of linear differential equations with irregular singular points. Proc. Japan Acad. **56A**, 97–102 (1980).

[3] I. M. Krichever: Russian Math. Surveys, **32**, 185–213 (1977).

[4] H. Flaschka and A. C. Newell: Monodromy and spectrum preserving deformations. I. Clarkson College of Tech. (1979) (preprint).

[5] W. Wasow: Asymptotic Expansions for Ordinary Differential Equations. N.Y. Interscience Pub. (1965).

[6] E. A. Coddington and N. Leivnson: Theory of Ordinary Differential Equations. McGraw-Hill (1955).

[7] M. Jimbo, T. Miwa, M. Sato, and Y. Môri: Holonomic quantum fields. The unanticipated link between deformation theory of differential equations and quantum fields. RIMS preprint, no. 305 (1979).

[8] K. Ueno: Monodromy preserving deformation and its application to soliton theory. II. RIMS preprint no. 309 (1980).

[9] K. Okamoto: Polynomial Hamiltonians associated with the Painlevé equations. Univ. of Tokyo (1980) (preprint).

[10] B. Klares: C. R. Acad. Sc. Paris, **288**, 205–208 (1979).