33. *Ultradifferentiability of Solutions of Ordinary Differential Equations*

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Let M_p, $p=0,1,2,\cdots$, be a sequence of positive numbers. An infinitely differentiable function f on an open set Ω in $\boldsymbol{R}^n$ is said to be an *ultradifferentiable function of class* $\{M_p\}$ (resp. *of class* (M_p)) if for each compact set K in Ω there are constants h and C (resp. and for each $h>0$ there is a constant C) such that

$$\sup_{x\in K}|D^\alpha f(x)|\leqq Ch^{|\alpha|}M_{|\alpha|}, \qquad |\alpha|=0,1,2,\cdots.$$

We assume that M_p satisfies the following conditions:

$$(1)\qquad M_0=M_1=1;$$

$$(2)\qquad (M_q/q\,!)^{1/(q-1)}\leqq(M_p/p\,!)^{1/(p-1)}, \qquad 2\leqq q\leqq p,$$

and furthermore in case of class (M_p)

$$(3)\qquad \left(\frac{M_p}{p\,!}\right)^2\leqq\left(\frac{M_{p-1}}{(p-1)\,!}\right)\left(\frac{M_{p+1}}{(p+1)\,!}\right), \qquad p=1,2,\cdots,$$

and

$$(4)\qquad M_p/(pM_{p-1})\to\infty \qquad \text{as } p\to\infty.$$

We consider the initial value problem of ordinary differential equation

$$(5)\qquad \begin{cases}\dfrac{dx}{dt}=f(t,x),\\ x(0)=y,\end{cases}$$

where $f(t,x)=(f_1,\cdots,f_n)$ is an n-tuple of functions defined on $(-T,T)\times\Omega$ with a $T>0$ and an open set Ω in $\boldsymbol{R}^n$. We assume the Lipschitz condition in x. Then for each relatively compact open subset Ω_1 of Ω there is a $0<T_1\leqq T$ such that (5) has for each $y\in\Omega_1$ a unique solution $x=x(t,y)$ on the interval $(-T_1,T_1)$.

Our main result is the following

Theorem. *If all components of $f(t,x)$ are ultradifferentiable functions of class $\{M_p\}$ (resp. of class (M_p)) on $(-T,T)\times\Omega$, then the components of the solution $x(t,y)$ are also ultradifferentiable functions of class $\{M_p\}$ (resp. of class (M_p)) on $(-T_1,T_1)\times\Omega_1$.*

Hereafter we denote by $*$ either $\{M_p\}$ or (M_p). The theorem is proved in two steps.

Proposition 1. *If $f(t,x)$ is ultradifferentiable of class $*$ only in x but uniformly in t, then $x(t,y)$ is ultradifferentiable of class $*$ in y uniformly in t.*

The proof in the case of class (M_p) is reduced to the case of class $\{M_p\}$ by Lemma 6 of [2]. Therefore we may restrict ourselves to the latter case.

We employ the method of Leray-Ohya [3] when they proved the ultradifferentiability of the Gevrey class $\{p!^s\}$ for solutions of hyperbolic equations.

Let

$$F(t,X)=\sum_{p=0}^{\infty}\frac{F_p(t)}{p!}X^p$$

be a formal power series in X with coefficients $F_p(t)$ which are functions in t. We write

$$f(t,x)\underset{\Omega}{\ll}F(t,X),\qquad t\in I, \tag{6}$$

if every component f_i of f satisfies

$$|D_x^\alpha f_i(t,x)|\leqq F_{|\alpha|}(t),\qquad x\in\Omega,\quad |\alpha|=0,1,2,\cdots,$$

for all $t\in I$. Let

$$\Phi(t,Y)=\sum_{q=0}^{\infty}\frac{\Phi_q(t)}{q!}Y^q\gg 0$$

be another formal power series in Y. Then we define

$$\bar{F}(t,\Phi(t,Y))=\sum_{p=0}^{\infty}\frac{F_p(t)}{p!}(n(\Phi(t,Y)-\Phi(t,0)))^p.$$

If $x(t,y)$ is an n-tuple of functions on $I\times\Omega_1$ with values in Ω such that

$$x(t,y)\underset{\Omega_1}{\ll}\Phi(t,Y),\qquad t\in I, \tag{7}$$

and if (6) holds, then we have

$$f(t,x(t,y))\underset{\Omega_1}{\ll}\bar{F}(t,\Phi(t,Y)),\qquad t\in I. \tag{8}$$

Lemma 1. *Suppose that* (6) *holds for* $I=[0,T_2]$. *If* $\Phi(t,Y)$ *satisfies*

$$\begin{cases}\dfrac{\partial\Phi(t,Y)}{\partial t}\gg\bar{F}(t,\Phi(t,Y)), & t\in I,\\ \Phi(0,Y)\gg Y,\end{cases} \tag{9}$$

then the solution $x(t,y)$ *of* (5) *is majorized as*

$$x(t,y)\underset{\Omega_1}{\ll}\Phi(t,Y),\qquad t\in I. \tag{10}$$

Proof. The solution $x(t,y)$ is obtained as the limit of Picard's approximation:

$$x_0(t,y)=y;$$

$$x_{k+1}(t,y)=y+\int_0^t f(s,x_k(s,y))ds.$$

Clearly we have

$$x_0(t,y)=y\underset{\Omega_1}{\ll}Y\ll\Phi(t,Y),\qquad t\in I.$$

Suppose that

$$x_k(t,y)\underset{\Omega_1}{\ll}\Phi(t,Y),\qquad t\in I.$$

Then we have

$$x_{k+1}(t,y)\underset{\Omega_1}{\ll} Y+\int_0^t \bar{F}(s,\Phi(s,Y))ds\ll\Phi(t,Y),\qquad t\in I.$$

Since $D_y^\alpha x_k(t,y)$ converges to $D_y^\alpha x(t,y)$, we have (10). The convergence itself may also be proved by the majorant method as above.

By shrinking $(-T,T)$ and Ω if necessary we can take

$$F(t,X)=C\sum_{p=0}^{\infty}\frac{M_p}{p\,!}\left(\frac{h}{n}X\right)^p \tag{11}$$

with constants h and C.

Suppose that $M_p=p\,!$. Then

$$\bar{F}(t,\Phi(t,Y))=\frac{C}{1-h(\Phi(t,Y)-\Phi(t,0))}.$$

Hence $\Phi(t,Y)$ is obtained as a solution of

$$\begin{cases}\dfrac{\partial\Phi(t,Y)}{\partial t}=\dfrac{C}{1+Cht-h\Phi(t,Y)},\\ \Phi(0,Y)=Y.\end{cases}\tag{12}$$

Since $\Phi(t,Y)$ is majorized for $t\geqq 0$ by the solution

$$\Psi(t,Y)=\frac{1}{h}-\sqrt{\left(\frac{1}{h}-Y\right)^2-\frac{2Ct}{h}}\tag{13}$$

of

$$\begin{cases}\dfrac{\partial\Psi(t,Y)}{\partial t}=\dfrac{C}{1-h\Psi(t,Y)},\\ \Psi(0,Y)=Y,\end{cases}\tag{14}$$

we can find for any $T_2<(2Ch)^{-1}$ constants k and B such that

$$\Phi_q(t)\leqq Bk^q q\,!,\quad 0\leqq t\leqq T_2,\quad q=0,1,2,\cdots.$$

In the general case we obtain a solution $\Phi(t,Y)$ of (9) by multiplying the coefficient of Y^p in the solution of (12) by $M^p/p\,!$, so that we have

$$\Phi_q(t)\leqq Bk^q M_q,\quad 0\leqq t\leqq T_2,\quad q=0,1,2,\cdots.\tag{15}$$

In fact, let $\varphi(t,Y)=\Phi(t,Y)-Ct$, where $\Phi(t,Y)$ is the solution of (12). Then it is the limit of Picard's approximation

$$\varphi_0(t,Y)=Y,$$

$$\varphi_{k+1}(t,Y)=Y+C\int_0^t\sum_{p=1}^{\infty}(h\varphi_k(s,Y))^p ds.$$

In general suppose that

$$\sum_{r=1}^{\infty}d_r(t)Y^r=\sum_{p=1}^{\infty}\left(h\sum_{q=1}^{\infty}c_q(t)Y^q\right)^p.$$

Then the coefficient

$$\sum_{p=1}^{r}\frac{M_p}{p\,!}h^p\sum_{q_1+\cdots+q_p=r}c_{q_1}(t)\frac{M_{q_1}}{q_1\,!}\cdots c_{q_p}(t)\frac{M_{q_p}}{q_p\,!}$$

of Y^r in

$$\sum_{p=1}^{\infty}\frac{M_p}{p!}\Big(h\sum_{q=1}^{\infty}c_q(t)\frac{M_q}{q!}Y^q\Big)^p$$

is less than or equal to $d_r(t)M_r/r!$ because it follows from (2) that

$$\frac{M_p}{p!}\frac{M_{q_1}}{q_1!}\cdots\frac{M_{q_p}}{q_p!}\leqq\frac{M_r}{r!}.$$

Therefore if we multiply the coefficient of Y^q in $\varphi_k(t,Y)$ by $M_q/q!$ and denote it again by $\varphi_k(t,Y)$, we have

$$\varphi_0(t,Y)=Y,$$

$$\varphi_{k+1}(t,Y)\gg Y+C\int_0^t\sum_{p=1}^{\infty}\frac{M_p}{p!}(h\varphi_k(s,Y))^p ds.$$

Hence $\Phi(t,Y)=\lim_{k\to\infty}\varphi_k(t,Y)+Ct$ satisfies (9).

In view of Lemma 1 the estimates (15) prove Proposition 1 for sufficiently small T_1. If $T_1>T_2$, we solve the equation with initial data at $t=T_2$. Since composites of ultradifferentiable mappings of class $*$ are ultradifferentiable of class $*$ under condition (2), we obtain Proposition after a finite number of repetitions.

The proof of the theorem will be completed if we show that a solution $x(t,y)$ of

$$\frac{dx}{dt}=f(t,x) \tag{16}$$

with parameters y is ultradifferentiable of class $*$ in t and y if it is ultradifferentiable in y uniformly in t.

Since the infinite differentiability in t and y is easy to prove, we need only to estimate $D_t^j D_y^\alpha x(t_0,y)$ for each fixed t_0. The formal Taylor expansion

$$x_{t_0}(t,y)=\sum_{j=0}^{\infty}\frac{\partial^j x(t_0,y)}{\partial t^j}\frac{(t-t_0)^j}{j!}$$

satisfies equation (16) as a formal power series in $t-t_0$ with infinitely differentiable functions of y as coefficients.

Thus the proof is reduced to the following proposition of the Cauchy-Kowalevsky type.

Proposition 2. *If a formal power series*

$$x_{t_0}(t,y)=\sum_{j=0}^{\infty}x^{(j)}(y)\frac{(t-t_0)^j}{j!}$$

in $t-t_0$ with C^∞ coefficients satisfies equation (16) *and if the initial value $x^{(0)}(y)$ is ultradifferentiable of class $*$ on Ω_1, then $x_{t_0}(t,y)$ is ultradifferentiable of class $*$ in the sense that for each compact set K in Ω_1 there are constants l and A (resp. and for each $l>0$ there is a constant A) such that*

$$\sup_{y\in K}|D_y^\alpha x^{(j)}(y)|\leqq Al^{j+|\alpha|}M_{j+|\alpha|},\qquad |\alpha|, j=0,1,2,\cdots.$$

The constants l and A (resp. constant A) depend only on the ultradifferentiability of $x^{(0)}(y)$ and are independent of t_0.

Again we may restrict ourselves to the case of class $\{M_p\}$.

Suppose that

$$f(t, x) \underset{\{t_0\}\times\Omega}{\ll} F(\bar{X}) = \sum_{p=0}^{\infty} \frac{F_p}{p!} \bar{X}^p$$

in the sense that

$$|D_t^j D_x^\alpha f_i(t_0, x)| \leqq F_{j+|\alpha|}, \qquad x \in \Omega, \quad j, |\alpha| = 0, 1, 2, \cdots,$$

and that

$$x_{t_0}(t, y) \underset{\{t_0\}\times\Omega_1}{\ll} \Phi(\bar{Y}).$$

Then we have

$$f(t, x(t, y)) \underset{\{t_0\}\times\Omega_1}{\ll} \bar{F}(\Phi(\bar{Y}))$$

$$= \sum_{p=0}^{\infty} \frac{F_p}{p!} (\bar{Y} + n(\Phi(\bar{Y}) - \Phi(0)))^p.$$

Hence we obtain the following lemma as in [2].

Lemma 2. *If*

(17) $$\frac{d\Phi(\bar{Y})}{d\bar{Y}} \gg \bar{F}(\Phi(\bar{Y})),$$

and

(18) $$\Phi(\bar{Y}) \underset{\Omega_1}{\gg} x^{(0)}(y),$$

then

(19) $$x_{t_0}(t, y) \underset{\{t_0\}\times\Omega_1}{\ll} \Phi(\bar{Y}).$$

In case $M_p = p!$ we can take $F(\bar{X}) = C(1 - h\bar{X})^{-1}$ with constants h and C. Therefore the equation for $\varphi(\bar{Y}) = \Phi(\bar{Y}) - \Phi(0) + \bar{Y}/n$ becomes

$$\begin{cases} \dfrac{d\varphi(\bar{Y})}{d\bar{Y}} = \dfrac{C}{1 - nh\varphi(\bar{Y})} + \dfrac{1}{n}, \\ \varphi(0) = 0. \end{cases}$$

In view of (13) the solution is majorized as

$$\frac{1}{nh}(1 - \sqrt{1 - 2nhC\bar{Y}}) \ll \varphi(\bar{Y}) \ll \frac{1}{nh}(1 - \sqrt{1 - 2nhC'\bar{Y}}),$$

where $C' = C + 1/n$. Hence if we take h and C sufficiently large,

$$\varphi(\bar{Y}) \gg \frac{Bk\bar{Y}}{1 - k\bar{Y}} + \frac{\bar{Y}}{n},$$

so that (18) holds. On the other hand (19) implies

$$x_{t_0}(t, y) \underset{\{t_0\}\times\Omega_1}{\ll} \frac{A}{1 - l\bar{Y}}$$

for some constants l and A.

The reduction of the general case to the above is similar to Proposition 1.

Combining Theorem with the implicit function theorem in [1], we obtain the Frobenius theorem for ultradifferentiable manifolds of class $*$.

References

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