27. Lévy's Functional Analysis in Terms of an Infinite Dimensional Brownian Motion. II

By Yoshihei HASEGAWA
Department of Mathematics, Nagoya Institute of Technology

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1980)

§1. **Introduction.** This note is a continuation of our previous note [2] and we shall use the terminologies in [2].

We shall consider various Dirichlet problems on the unit ball D_∞ in the space $\boldsymbol{E}$ defined in [2, §3]. In particular we shall establish a finite dimensional approximation theorem (Theorem 4.1) of Dirichlet solutions on D_∞ which may be regarded as a reformulation of Lévy's "la méthode du passage du fini à l'infini" (see [1, p. 307]).

§2. **Spherical Brownian motion** (continued from [2, §4]). The standard Gaussian white noise μ defined in [2, §2] can be easily extended to the measurable space $(S_\infty, \mathcal{S}_\infty)$ as follows:

$$\mu(A)=P^0(B(1,\omega)\in A) \qquad \text{for } A\in\mathcal{S}_\infty,$$

where $\{B(t)\}$ is the Brownian motion given in [2, §3].

Our first assertion is

Theorem 2.1. *Let $f(\xi)$ be a bounded, cylindrically measurable, O_1-continuous function on S_∞. Then we have*

$$\lim_{t\to\infty}\tilde{E}^\xi[f(\xi_t)]=\int f(\zeta)\mu(d\zeta) \qquad \textit{for any } \xi\in S_\infty,$$

and the white noise μ is the unique invariant probability measure of the spherical Brownian motion $\{\xi_t\}$.

Consequently we have the following contraction semi-group $\{T_t : t\geqslant 0\}$ ([5, Chap. IX]) on the complex Hilbert space $L^2(S_\infty,\mu)$:

$$T_t f(\xi)=\tilde{E}^\xi[f(\xi_t)] \quad (t\geqslant 0) \qquad \text{for } f\in L^2(S_\infty,\mu).$$

Now we have

Theorem 2.2. *The infinitesimal generator of $\{T_t\}$ is a self-adjoint operator with the pure-point spectrum $\{-n/2\,; n=0,1,2,\cdots\}$ and the eigenspace of the eigenvalue $-n/2$ is spanned by $\{\Xi_K\,; |K|=n\}$,* (see [2, §2] for definitions). *This infinitesimal generator agrees with the infinite dimensional Laplacian operator of Y. Umemura* (see [4]), *up to constant* 1/2.

Next we shall see that the spherical Brownian motion is homogeneous under the group G of linear bimeasurable bijections g of $\boldsymbol{E}$ to $\boldsymbol{E}$ satisfying

$$\mu(g\,\cdot)=\mu(\cdot) \quad \text{and} \quad \|gx\|_\infty=\|x\|_\infty \quad \text{for } x\in\boldsymbol{E}.$$

Proposition 2.3. *For $g\in G$ and $A\in\mathcal{S}_\infty$, it holds that*

$$\tilde{P}^{g\xi}(\xi_t \in gA) = \tilde{P}^{\xi}(\xi_t \in A).$$

For a given sequence of integers $(p_1, \cdots, p_n, \cdots)$ such that

$$2 < p_1 < p_2 < \cdots < p_n < \cdots, \quad \text{and} \quad \lim_{n\to\infty} p_{n+1}/p_n = 1,$$

we are given a bijection σ of $\{1, 2, 3, \cdots\}$ onto itself which is also a permutation of $\{p_n+1, p_n+2, \cdots, p_{n+1}\}$ for each n. We then define the mapping g_σ of $\boldsymbol{E}$ onto itself by

$$(g_\sigma x)_n = x_{\sigma(n)} \qquad \text{for } x = (x_1, x_2, x_3, \cdots) \in \boldsymbol{E}.$$

Theorem 2.4. *Mappings g_σ of the above-mentioned type form a subgroup G_0 of G.*

§3. Dirichlet problems on the unit ball of $\boldsymbol{E}$. We denote by τ_U and τ the first exit time from a domain U of the Brownian motion $\boldsymbol{B}$, and the one from the unit ball $D_\infty = \{x \in \boldsymbol{E}; \|x\|_\infty < 1\}$ respectively.

Now we reformulate the Dirichlet problems treated by Paul Lévy in his book [1] in terms of the infinite dimensional Brownian motion.

Theorem 3.1. *For a real bounded measurable continuous function $\phi(\xi)$ on S_∞, the function f defined on D_∞ by*

$$f(x) = E^x[\phi(B_\tau)]$$

is the unique function satisfying the following conditions:

i) *The function $f(x)$ is a real bounded measurable continuous function on D_∞.*

ii) *For any subdomain U such that the exit time τ_U from U is a stopping time and the closure $\bar{U}$ of U is in D_∞, it holds that*

$$f(x) = E^x[f(B(\tau_U))] \quad \text{on} \quad D_\infty.$$

iii) *For any point $\xi \in S_\infty$,*

$$\lim_{x\to\xi} f(x) = \phi(\xi).$$

Theorem 3.2. *For $e_1, e_2, \cdots, e_p \in \boldsymbol{E}$ such that $\|e_j\|_\infty > 0$ and for a real bounded measurable function $h(s_1, \cdots, s_p, x_1, \cdots, x_m)$ on the space $R_+^p \times R^m$ ($R_+ = [0, \infty)$), we define the function $\phi(\xi)$ on S_∞ such that*

$$\phi(\xi) = h(\|\xi + e_1\|_\infty, \cdots, \|\xi + e_p\|_\infty, \xi_1, \cdots, \xi_m)$$

for $\xi = (\xi_1, \cdots, \xi_n, \cdots) \in S_\infty$. Then the function f defined on D_∞ by

$$f(x) = E^x[\phi(B_\tau)]$$

is the unique function satisfying the following conditions:

i) *The function $f(x)$ is a real bounded measurable function on D_∞.*

ii) *For any $x \in D_\infty$, the function $f(B_t(\omega))$ is continuous in t on $[0, \tau(\omega))$ a.s. P^x, and*

$$\lim_{t\uparrow\tau(\omega)} f(B_t(\omega)) = \phi(B_\tau(\omega)) \quad a.s. \quad P^x.$$

iii) *For any domain U such that the exit time τ_U is a stopping time and $\bar{U} \subset D_\infty$, it holds that*

$$f(x) = E^x[f(B(\tau_U))] \quad \text{on} \quad D_\infty.$$

Theorem 3.3. *Let $\phi(\xi)$ be a tame function on S_∞ given by*

$$\phi(\xi) = \tilde{\phi}(\xi_1, \cdots, \xi_m) \qquad \text{for } \xi = (\xi_1, \cdots, \xi_m, \cdots) \in S_\infty,$$

where $\tilde{\phi}(u_1, \cdots, u_m)$ *is a real measurable function on* R^m *such that*

$$\int \tilde{\phi}(u)^2 \exp(-u^2/2)du < \infty.$$

Then the function f *defined on* D_∞ *by*

$$f(x) = E^x[\phi(B_\tau)]$$

is the unique function satisfying the following conditions:

i) *The function* $f(x)$ *is a real measurable continuous function on* D_∞.

ii) *For any subdomain* U *such that the exit time* τ_U *is a stopping time and* $\bar{U} \subset D_\infty$, *it holds that*

$$f(x) = E^x[f(B(\tau_U))] \quad \text{on} \quad D_\infty.$$

iii) $\lim_{t \uparrow \tau(\omega)} f(B_t(\omega)) = \phi(B_\tau(\omega))$ *a.s.* P^x, $x \in D_\infty$.

iv) *For any point* $x \in D_\infty$ *and for any sequence of subdomains* $\{U_n\}$ *such that each exit time* $\tau_n = \tau_{U_n}$ *is a stopping time and* $\bar{U}_n \subset U_{n+1}$, $\bigcup_{n=1}^{\infty} U_n = D_\infty$, *the family* $\{f(B(\tau_n)); n = 1, 2, 3, \cdots\}$ *is uniformly* P^x-*integrable.*

Next we define the *harmonic measure* μ_x relative to $x \in D_\infty$ and D_∞ by putting

$$\mu_x(A) = P^x(B_\tau \in A) \qquad \text{for } A \in \mathcal{S}_\infty.$$

Then we have

Theorem 3.4. i) *Harmonic measures* μ_x *and* μ_y *are equivalent, if and only if the sequence* $x - y = (x_1 - y_1, \cdots, x_n - y_n, \cdots)$ *is square summable.*

ii) *For any* $g \in G$, $x \in D_\infty$ *and* $A \in \mathcal{S}_\infty$, *it holds that*

$$\mu_{gx}(gA) = \mu_x(A).$$

Here we pause to give an interpretation to the Lévy's mean value formula (see [1, p. 316, (24)]). Take $a \in D_\infty$ and r such that $0 < 2r < 1 - \|a\|_\infty$ and denote by $S_\infty(a, r)$ the sphere $\{x \in \boldsymbol{E}; \|x - a\|_\infty = r\}$. Define a mapping $T: S_\infty \to S_\infty(a, r)$ such that

$$T\xi = (1 - (r/\|\xi - a\|_\infty))a + (r/\|\xi - a\|_\infty)\xi.$$

Then we have

Theorem 3.5. *For any measurable subset* $A \subset S_\infty(a, r)$,

$$\mu_a(T\xi \in A) = \mu((A - a)/r)$$

with the standard Gaussian white noise μ.

A domain D is said to be *semi-bounded*, if $\{\|x\|_n; x \in D\}$ is bounded in R for a semi-norm $\|\cdot\|_n$ $(1 \leqslant n \leqslant \infty)$. Now we say a measurable function f on a domain D to be *harmonic*, if the following conditions are satisfied:

i) For any semi-bounded domain U such that τ_U is a stopping time and $\bar{U} \subset D$, it holds that

$$f(x) = E^x[f(B(\tau_U))] \quad \text{on} \quad D.$$

ii) The function $f(B_t(\omega))$ is continuous in t on $[0, \tau(\omega))$ a.s. P^x,

$x \in D$, where $\tau(\omega)$ is the exit time from D.

As a pathological phenomenon, the unicity principle does not hold. In fact we can prove

Proposition 3.6. *For a real continuous bounded function ϕ on R and $\xi \in \boldsymbol{E}$ such that $\|\xi\|_\infty > 0$, we define the function $f(x)$ on $\boldsymbol{E}$ by*

$$f(x) = \phi(\langle x, \xi\rangle_\infty),$$

where $4\langle x, \xi\rangle_\infty = \|x+\xi\|_\infty^2 - \|x-\xi\|_\infty^2$. Then the function $f(x)$ is harmonic on $\boldsymbol{E}$.

Further we have

Proposition 3.7. *Let $f(x)$ be a tame function on $\boldsymbol{E}$ given by*

$$f(x) = \phi(x_1, \cdots, x_n) \qquad \text{for } x = (x_1, \cdots, x_n, \cdots) \in \boldsymbol{E},$$

where $\phi(x_1, \cdots, x_n)$ is a harmonic polynomial on R^n. Then the function $f(x)$ on $\boldsymbol{E}$ is harmonic on $\boldsymbol{E}$.

§4. Finite dimensional approximation to Dirichlet solutions on the unit ball. In the previous sections, we defined the space $\boldsymbol{E}$, the O_1-topology and the Brownian motion $\boldsymbol{B}$ on $\boldsymbol{E}$. We are going to show that these objects are compatible in the sense mentioned-below.

With a real function $\tilde{\phi}(u_1, \cdots, u_m)$ on R^m such that

$$\int \tilde{\phi}(u)^2 \exp(-u^2/2) du < \infty, \tag{4.1}$$

we associate the function $\phi(\xi)$ on S_∞

$$\phi(\xi) = \tilde{\phi}(\xi_1, \cdots, \xi_m).$$

Now we note that the projection ρ_n which has been introduced by using the branching rule in [2, §2] can be regarded as a projection from $L^2(S_\infty, \mu)$ to $L^2(S_n, \mu_n)$, where S_n is the n-dimensional sphere with radius $\sqrt{n+1}$ and μ_n is the uniform probability measure on S_n. Hence we obtain a projectively consistent sequence $\{\rho_n\phi; n \geqslant 1\}$ and we can see that each $\rho_n\phi$ has a continuous version ϕ_n on S_n. Let $f_n(x)$ be the Dirichlet solution on the ball $D_n = \{(x_1, \cdots, x_{n+1}); x_1^2 + \cdots + x_{n+1}^2 < n+1\}$ corresponding to the boundary function ϕ_n. Now let us lift $f_n(x)$ to a function $f_n(\pi_n x)$ on $\{x \in D_\infty; \|x\|_\infty > 0\}$ by the following projection π_n:

$$\pi_n x = (\|x\|_\infty / \|x\|_{n+1})(x_1, \cdots, x_{n+1}) \in R^{n+1} \setminus \{0\}$$

for $x = (x_1, \cdots, x_{n+1}, \cdots) \in \boldsymbol{E}$ with $\|x\|_\infty > 0$, $\|x\|_{n+1}$. Then we have the following remarkable result.

Theorem 4.1. *If the function $\tilde{\phi}$ satisfies the condition* (4.1) *and an additional condition:*

$$\int |\tilde{\phi}(u)| du < \infty, \tag{4.2}$$

then for any point $x \in D_\infty$ such that $\lim_{n\to\infty} \|x\|_n = \|x\|_\infty > 0$,

$$\lim_{n\to\infty} f_n(\pi_n x) = E^x[\phi(B_\tau)]. \tag{4.3}$$

Further we can show that the equality (4.3) *holds for polynomials $\tilde{\phi}(u_1, \cdots, u_m)$, which do not satisfy the condition* (4.2).

References

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