15. *Piecewise Linear Dehn's Lemma in 4-Dimensions*

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§1. **Statement of result.** Let W be a compact 4-manifold with non-empty boundary $\partial W = M$. Throughout this note, we shall adopt the convention that the handle decomposition of W possesses no 4-handles and only one 0-handle.

In [4], Norman has given a number of cases for which the analogue of Dehn's lemma in 4-dimensions works. See also Fenn [2]. Some examples for which such an analogue fails can be seen in [1] and [3]. Our version of Dehn's lemma is as follows:

Theorem. *Let W be a compact 4-manifold with non-empty boundary $\partial W = M$, and let $h: (D^2, S^1) \to (W, M)$ be a proper map whose restriction to S^1 is an embedding. Suppose that W admits a handle decomposition without 1-handles (hence is simply-connected), then h is homotopic to a PL-embedding keeping the boundary fixed. In particular, every loop on M bounds a PL-disc in W.*

§2. **Key lemma.** For a compact 4-manifold N with boundary $\partial N = M \cup M'$ (disjoint), the triad $(N; M, M')$ always admits a self-indexing Morse function $f: N \to [0, 4]$ such that $f^{-1}(0) = M$ and $f^{-1}(4) = M'$. Let D be a proper smooth submanifold of N of codimension two.

Lemma. *D can be isotopically deformed in N keeping the boundary fixed so that it satisfies the following*

(1) *$g = f \mid D: D \to [0, 4]$ is also a Morse function.*

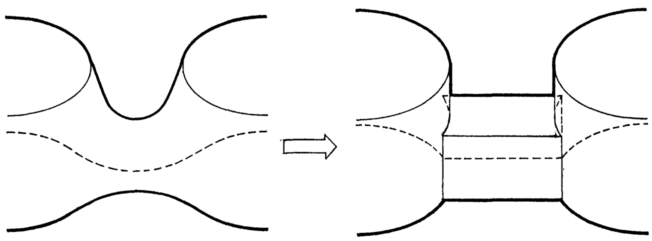
(2) *For a critical point of g of index 1, its critical value is less than 2.*

Proof. The condition (1) is attained by small perturbation of D, because it is a smooth submanifold. Then, one may assume that there are no critical points of f on D, and no critical value of g is $1, 2$ or 3. Next, push down (or push up) a small neighborhood on D of each critical point of g of index 0 (or index 2) so that its critical value turns out to be less than 1 (or greater than 3). Then, it follows from the general position lemma that the same procedure as above makes the critical value of each critical point of g of index 1 less than 3.

Take a gradient like vector field of f on N, which defines the core of a 2-handle of N in $f^{-1}([a, 2])$ for some a $(1 < a < 2)$. This also defines the co-core of it in $f^{-1}([2, b])$ for some b $(2 < b < 3)$. Let C denote the union of cores and co-cores of all 2-handles of N in $f^{-1}([a, b])$. Then, $S = C \cap f^{-1}(b)$ is the union of all right hand attaching spheres on $f^{-1}(b)$. By taking real numbers a, b closely to 2 if necessary, one can make C and disjoint.

Let c, d be real numbers with $2<b<c<d<3$. Then, deforming D isotopically so as to arrange critical levels of g in the situation above, one may assume that the critical value of each critical point of g of index 1 is either c or less than a, and that there are no critical points of index 0 and 2 on $g^{-1}([a, d])$.

Now, make a corner on D as in the figure below.



This device gives the trace of the following change of links: the link $L' = D \cap f^{-1}(d)$ is modified by band connected sums at the level c and a new link $L = D \cap f^{-1}(b)$ is obtained. Since f has no critical levels in $[b, c]$, there exists a trivialization

$$T : f^{-1}(c) \times [b, c] \rightarrow f^{-1}([b, c]).$$

Let $B = I \times J$ denote one of the bands in $f^{-1}(c)$, where I, J are closed intervals and $T(B \times \{b\}) \cap L = T(I \times \partial J \times \{b\})$. In case $T(B \times \{b\})$ does not intersect S, deform D isotopically to $(D - T(\partial I \times J \times [b, c] \cup B \times \{c\})) \cup T(I \times \partial J \times [b, c] \cup B \times \{b\})$. This makes the critical value ($=c$) corresponding to B come down to b. Moreover, one can push down the band further to the level a, because $T(B \times \{b\}) \cap S = \phi$ and $f^{-1}([a, b]) - C$ is diffeomorphic to $(f^{-1}(c) - S) \times [a, b]$. For general case, deform D similarly as above using the shortened band $B' = I \times J'$, instead of B, where $J' \subset J$ is a closed interval so that $T(I \times J' \times \{b\})$ does not intersect S. Doing this procedure for each band and smoothing corners again, we finally obtain the required isotopy of D satisfying the conditions (1), (2). This completes the proof.

§3. **Proof of Theorem.** By general position, h is homotopic to a proper immersion $h' : (D^2, S^1) \rightarrow (W, M)$ which has only double point singularities in *int* D^2. One can choose this homotopy so that the boundary remains fixed. Take a small closed ball neighborhood of each singular point and connect them up by thin tubes in $W - h'(D^2)$. Since the resulting 4-ball B^4 contains all the singularities of h', $D = h'(D^2) \cap (W - \mathit{int}\, B^4)$ is a connected, smooth submanifold.

We apply Lemma to the triad $(W - \mathit{int}\, B^4 ; \partial B^4, M)$ and the submanifold D. So, for a self-indexing Morse function $f: W-int\, B^4 \to [0, 4]$, D will be isotopically deformed so that $g=f|D$ is also a Morse function and it has no critical points of index 0 and 1 on $g^{-1}([e, 4])$. where e is any number with $1<a<e<2$. Then $D\cap f^{-1}(e)$ is the link of q components in $f^{-1}(e)$. Choose $(q-1)$ bands $B_1, B_2, \cdots, B_{q-1}$ which connect those q circles in $f^{-1}(e)$ with compatible orientations. Then $A=D\cap f^{-1}([e, 4])\cup B_1\cup \cdots \cup B_{q-1}$ turns out to be an annulus and one of its boundary defines a knot K in $f^{-1}(e)$.

Now, since W admits a handle decomposition without 1-handles, a Morse function f can be chosen so that it has no critical points of index 1. Then, $f^{-1}(e)$ is in fact diffeomorphic to the 3-sphere and $B^4\cup f^{-1}([0, e])$ is diffeomorphic to the 4-disc. Hence, attaching a cone $p*K$ from the center p of this 4-disc to the annulus A, one obtains a properly PL-embedded 2-disc. It can be identified with the image of some PL-embedding of a 2-disc homotopic to h keeping the boundary fixed. This completes the proof.

References

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[4] Norman, R. A.: Dehn's lemma for certain 4-manifolds. Inventiones Math., **7**, 143–147 (1969).