95. *Some Lie Algebras of Vector Fields on Foliated Manifolds and their Derivation Algebras*

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1. We want to define some structures on foliated manifolds and Lie algebras of vector fields associated with the structures, and determine their derivation algebras. We have two directions: One is to consider structures on leaves; the other on transversals to leaves. In this article we treat only the former (see [2] for details and proofs), and for the latter we will discuss elsewhere.

Let M be a $(p+q)$-dimensional smooth manifold, and $\mathcal{F}$ a codimension q foliation on M. Denote by $\mathcal{T}(M,\mathcal{F})$ the Lie algebra of all leaf-tangent vector fields on $(M,\mathcal{F})$, and by $\Omega(M)$ the exterior algebra of all differential forms on M, and define its differential ideal $\mathcal{I}(M,\mathcal{F})$ as

$$\begin{aligned}\mathcal{I}(M,\mathcal{F})&=\{\alpha\in\Omega(M)\,;\,\alpha(X_1,X_2,\cdots)=0 \text{ for } X_i\in\mathcal{T}(M,\mathcal{F})\}\\&=\{\alpha\in\Omega(M)\,;\,\iota_L^*\alpha=0 \text{ for every leaf } L \text{ of } \mathcal{F}\},\end{aligned}$$

where ι_L is the inclusion mapping of L in M. Then $\mathcal{I}(M,\mathcal{F})$ is L_X-stable for any $X\in\mathcal{T}(M,\mathcal{F})$, where L_X means the Lie derivative.

A p-form τ on M is called a partially unimodular structure on $(M,\mathcal{F})$, if $\iota_L^*\tau\neq 0$ for every leaf L of $\mathcal{F}$, that is, $\iota_L^*\tau$ is a volume form on L. Then τ is partially closed, that is, $d\tau\in\mathcal{I}(M,\mathcal{F})$.

Let $p=2n$. A 2-form ω on M is called a partially symplectic structure on $(M,\mathcal{F})$, if ω is partially closed and $\iota_L^*\omega$ is of rank $2n$ for every leaf L of $\mathcal{F}$.

Let $p=2n+1$. A 1-form θ on M is called a partially contact structure on $(M,\mathcal{F})$, if $(\iota_L^*\theta)\wedge(\iota_L^*d\theta)^n\neq 0$ for every leaf L of $\mathcal{F}$.

We can get normal forms of these partially classical structures on $(M,\mathcal{F})$ as follows; for suitable distinguished coordinates $(v_1,\cdots,v_p, w_1,\cdots,w_q)$

$$\tau\equiv dv_1\wedge\cdots\wedge dv_p,\quad \omega\equiv\sum_{i=1}^n dv_i\wedge dv_{i+n},\quad \theta\equiv dv_{2n+1}-\sum_{i=1}^n v_{i+n}dv_i \pmod{\mathcal{I}(M,\mathcal{F})}.$$

2. Let τ be a partially unimodular structure on $(M,\mathcal{F})$. A vector field $X\in\mathcal{T}(M,\mathcal{F})$ is called partially conformally unimodular, if $L_X\tau$ is congruent to $\phi\tau$ modulo $\mathcal{I}(M,\mathcal{F})$ for some function $\phi\in C^\infty(M)^{\mathcal{F}}$, where $C^\infty(M)^{\mathcal{F}}$ is the space of smooth functions on M which are constant on each leaves of $\mathcal{F}$. Moreover, if the function ϕ is zero, X is called partially unimodular. Then we get two natural Lie subalgebras of

$\mathfrak{X}(M, \mathcal{F})$:

$$\mathfrak{X}_\tau(M, \mathcal{F})=\{X \in \mathfrak{X}(M, \mathcal{F})\,;\, L_X\tau \in \mathcal{J}(M, \mathcal{F})\},$$

$$\mathfrak{X}_{c\tau}(M, \mathcal{F})=\{X \in \mathfrak{X}(M, \mathcal{F})\,;\, L_X\tau \equiv \phi\tau \ (\mathrm{mod}\, \mathcal{J}(M, \mathcal{F})) \text{ for some } \phi \in C^\infty(M)^{\mathcal{F}}\}.$$

We get the same Lie algebras for another partially unimodular structure congruent to τ modulo $\mathcal{J}(M, \mathcal{F})$.

For a partially symplectic structure ω, we can similarly define partially symplectic, and partially conformally symplectic vector fields, and get two Lie algebras

$$\mathfrak{X}_\omega(M, \mathcal{F})=\{X \in \mathfrak{X}(M, \mathcal{F})\,;\, L_X\omega \in \mathcal{J}(M, \mathcal{F})\},$$

$$\mathfrak{X}_{c\omega}(M, \mathcal{F})=\{X \in \mathfrak{X}(M, \mathcal{F})\,;\, L_X\omega \equiv \phi\omega \ (\mathrm{mod}\, \mathcal{J}(M, \mathcal{F})) \text{ for some } \phi \in C^\infty(M)^{\mathcal{F}}\}.$$

Let θ be a partially contact structure on $(M, \mathcal{F})$. A vector field $X \in \mathfrak{X}(M, \mathcal{F})$ is called partially contact, if $L_X\theta$ is congruent to $\phi\theta$ modulo $\mathcal{J}(M, \mathcal{F})$ for some function $\phi \in C^\infty(M)$. Such vector fields form the Lie algebra $\mathfrak{X}_\theta(M, \mathcal{F})$.

These Lie algebras $\mathfrak{X}(M,\mathcal{F})$, $\mathfrak{X}_\tau(M,\mathcal{F})$, $\mathfrak{X}_{c\tau}(M, \mathcal{F})$, $\mathfrak{X}_\omega(M, \mathcal{F})$, $\mathfrak{X}_{c\omega}(M, \mathcal{F})$ and $\mathfrak{X}_\theta(M, \mathcal{F})$ are called of partially classical type, and correspond in the formal case to É. Cartan's classification of "intransitive Lie algebras whose transitive parts are primitive and infinite" (see [3]).

3. From [1], the derivation algebra of $\mathfrak{X}(M, \mathcal{F})$ is naturally isomorphic to the Lie algebra $\mathcal{L}(M, \mathcal{F})$ of all locally foliation-preserving vector fields on M. Similarly we get the following Lie algebras;

$$\mathcal{L}_\tau(M, \mathcal{F})=\{X \in \mathcal{L}(M, \mathcal{F})\,;\, L_X\tau \in \mathcal{J}(M, \mathcal{F})\},$$

$$\mathcal{L}_{c\tau}(M, \mathcal{F})=\{X \in \mathcal{L}(M, \mathcal{F})\,;\, L_X\tau \equiv \phi\tau \ (\mathrm{mod}\, \mathcal{J}(M, \mathcal{F})) \text{ for some } \phi \in C^\infty(M)^{\mathcal{F}}\},$$

$$\mathcal{L}_\omega(M, \mathcal{F})=\{X \in \mathcal{L}(M, \mathcal{F})\,;\, L_X\omega \in \mathcal{J}(M, \mathcal{F})\},$$

$$\mathcal{L}_{c\omega}(M, \mathcal{F})=\{X \in \mathcal{L}(M, \mathcal{F})\,;\, L_X\omega \equiv \phi\omega \ (\mathrm{mod}\, \mathcal{J}(M, \mathcal{F})) \text{ for some } \phi \in C^\infty(M)^{\mathcal{F}}\},$$

$$\mathcal{L}_\theta(M, \mathcal{F})=\{X \in \mathcal{L}(M, \mathcal{F})\,;\, L_X\theta \equiv \phi\theta \ (\mathrm{mod}\, \mathcal{J}(M, \mathcal{F})) \text{ for some } \phi \in C^\infty(M)\}.$$

Then we get

Theorem. *Assume that a foliated manifold $(M, \mathcal{F})$ is equipped with a partially classical structure τ, ω or θ.*

(a) *Let $\sigma = c\tau$ $(p \neq 1)$, $c\omega$ or θ. Then*

$$H^1(\mathcal{L}_\sigma(M, \mathcal{F})\,;\, \mathcal{L}_\sigma(M, \mathcal{F}))=0,$$

$$H^1(\mathfrak{X}_\sigma(M, \mathcal{F})\,;\, \mathfrak{X}_\sigma(M, \mathcal{F})) \cong \mathcal{L}_\sigma(M, \mathcal{F})/\mathfrak{X}_\sigma(M, \mathcal{F}).$$

(b) *Let $\sigma=\tau$ $(p \neq 1)$ or ω. Then*

$$H^1(\mathcal{L}_\sigma(M, \mathcal{F})\,;\, \mathcal{L}_\sigma(M, \mathcal{F})) \cong \mathcal{L}_{c\sigma}(M, \mathcal{F})/\mathcal{L}_\sigma(M, \mathcal{F}),$$

$$H^1(\mathfrak{X}_\sigma(M, \mathcal{F})\,;\, \mathfrak{X}_\sigma(M, \mathcal{F})) \cong \mathcal{L}_{c\sigma}(M, \mathcal{F})/\mathfrak{X}_\sigma(M, \mathcal{F}).$$

Moreover, if $C^\infty(M)^{\mathcal{F}} \cong \boldsymbol{R}$ and is σ not partially exact, then

$$H^1(\mathcal{L}_\sigma(M, \mathcal{F})\,;\, \mathcal{L}_\sigma(M, \mathcal{F}))=0,$$

$$H^1(\mathfrak{X}_\sigma(M, \mathcal{F})\,;\, \mathfrak{X}_\sigma(M, \mathcal{F})) \cong \mathcal{L}_\sigma(M, \mathcal{F})/\mathfrak{X}_\sigma(M, \mathcal{F}).$$

Here σ is partially exact, if there exists a form α on M such that $\sigma \equiv d\alpha$ modulo $\mathcal{I}(M, \mathcal{F})$.

Recall that for any Lie algebra $\mathcal{L}$, $H^1(\mathcal{L}; \mathcal{L})$ is the quotient space of the derivation algebra of $\mathcal{L}$ modulo its ideal of inner derivations of $\mathcal{L}$. The proof of the theorem consists of two parts. One part is the proof of the theorem for the case (flat case) where M is an Euclidean space V and $\mathcal{F}$ is a standard foliation by parallel p-planes in V. The essential tool which we use here is the grading of the subalgebras of $\mathcal{T}_\sigma(V, \mathcal{F})$ and $\mathcal{L}_\sigma(V, \mathcal{F})$, consisting of vector fields with polynomial coefficients with respect to a fixed coordinates in V. The other part of the proof is the localization. Here we use essentially the fact that every derivation of $\mathcal{T}_\tau(M, \mathcal{F})$, $\mathcal{T}_\omega(M, \mathcal{F})$ or $\mathcal{T}_\theta(M, \mathcal{F})$ is localizable.

4. The case where $p=1$ and $\sigma=\tau$ or $c\tau$ is pathological, because $\mathcal{T}_\tau(M, \mathcal{F})$ is abelian and its derivations are not localizable. But for flat case, we can show Theorem (a) for $\mathcal{T}_{c\tau}(V, \mathcal{F})$ and $\mathcal{L}_{c\tau}(V, \mathcal{F})$.

We must remark the phenomena that there are derivations of $\mathcal{T}_\tau(V, \mathcal{F})$ and $\mathcal{L}_\tau(V, \mathcal{F})$ ($p=1$) which cannot be realizable by vector fields.

References

[1] Y. Kanie: Cohomologies of Lie algebras of vector fields with coefficients in adjoint representations: Foliated case. Publ. RIMS, Kyoto Univ., **14**, 487–501 (1978).

[2] ——: Some Lie algebras of vector fields and their derivations: Case of partially classical type (to appear).

[3] T. Morimoto: On the intransitive Lie algebras whose transitive parts are infinite and primitive, J. Math. Soc. Japan, **29**, 35–65 (1977).