56. A Note on the Blowup-Nonblowup Problems in Nonlinear Parabolic Equations

By Nobutoshi ITAYA
Kobe University of Commerce

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1. Many studies have been made on the following type of semilinear parabolic equations

$$\frac{\partial}{\partial t}u(x,t)=\Delta u+F(x,t,u), \tag{1.1}$$

especially in connection with the so-called blowing-up problems (cf. [2]). However, hardly any discussion has yet been made on the effect of the coefficient of Δu, as it is a function in u, upon the global behaviour of the solutions. Here, we discuss some subjects in this field. For simplicity, we restrict the spatial dimension to 1, which is not quite essential.

For an interval $I\subset R^1$ and $\alpha\in(0,1)$ we define $H^{2+\alpha}(\infty,I)$ as follows:

$$\begin{cases} H^{2+\alpha}(\infty,I)\equiv\{u(x,t), \text{ defined on } I\times[0,\infty); \text{ for } \forall T\in(0,\infty),\ u, \\ \text{as restricted to } I\times[0,T], \text{ belongs to } H^{2+\alpha}_{T(I)}\}, \end{cases} \tag{1.2}$$

where $H^{2+\alpha}_{T(I)}$ is a Hölder space defined by replacing R^1 by I in the definition of $H^{2+\alpha}_T$ (cf. [3]).

2.1. We consider the following mixed problem for a non-linear parabolic equation:

$$\frac{\partial}{\partial t}u(x,t)=\varphi(u)\frac{\partial^2}{\partial x^2}u+\psi(u),\ (x\in I\equiv[0,a]\ (a>0),\ t\geqq 0), \tag{2.1}$$

$$\begin{cases} u(x,0)=u_0(x)\in H^{2+\alpha}_{(I)}(\geqq 0),\ u(0,t)=u(a,t)=0(t\geqq 0), \\ u_0(0)=u_0(a)=u_0''(0)=u_0''(a)=0, \end{cases} \tag{2.2}$$

where $\varphi(u)$ and $\psi(u)$ are defined on $[0,\infty)$, and are monotonically increasing, non-negative, of the C^1-class on $[0,\infty)$ and of the C^2-class on $(0,\infty)$, and especially $\varphi(0)$ is positive. Without proof we state:

Theorem 2.1 (cf. [7] etc.). *For some $T\in(0,\infty)$, there exists a unique solution $u(x,t)$ for (2.1)–(2.2) belonging to $H^{2+\alpha}_{T(I)}$. (Note that $u(x,t)$ is non-negative.)*

We shall state below that, under some conditions on $\varphi(u)$, $\psi(u)$, and u_0, there is a unique solution $u(x,t)$ for (2.1)–(2.2) belonging to $H^{2+\alpha}(\infty,I)$, and that, under some other conditions on them, the solution $u(x,t)$ blows up in a finite time. We remark (cf. [3], [4], etc.) that, in order to show the former, we need only to have *a priori* estimates for $u(x,t)$ such that $|u|^{(0)}_{T(I)}\leqq A(T)(\nearrow(T\nearrow\infty))$, under the assumption that

$u\in H^{2+\alpha}_{T(I)}$ satisfies (2.1)–(2.2). We set $B(u)\equiv\psi(u)/\varphi(u)$ and $\bar{B}(u)\equiv\sup\{B(u'):0\leqq u'\leqq u\}$.

Theorem 2.2. *For the mixed problem* (2.1)–(2.2), *if* $B(u)$ *is bounded, then there exists a unique solution* $u(x,t)$ *belonging to* $H^{2+\alpha}(\infty,I)$. *Moreover, it holds that*

$$(2.3)\qquad |u|^{(0)}_{T(I)}\leqq|u_0|^{(0)}_{(I)}+\frac{K}{8}\cdot a^2\qquad(K\equiv\sup B(u)<\infty).$$

Proof. Take optionally $T\in(0,\infty)$. Assume that $u(x,t)\in H^{2+\alpha}_{T(I)}$ satisfies (2.1)–(2.2). Then $U(x,t)\equiv u(x,t)+\frac{K}{2}\cdot\left(x-\frac{a}{2}\right)^2$ satisfies

$$(2.4)\qquad\begin{cases}U_t=\varphi(u)U_{xx}+(B(u)-K)\varphi(u),\\ U(x,0)=U_0(x)\equiv u_0(x)+\frac{K}{2}\left(x-\frac{a}{2}\right)^2(\geqq0),\\ U(0,t)=U(a,t)=U_0(0)=U_0(a)=\frac{K}{8}a^2,\ U_0''(0)=U_0''(a).\end{cases}$$

By the condition on $B(u)$, the comparison theorem, and the definition of $U(x,t)$, we have the estimate (2.3). Q.E.D.

In case that $B(u)$ is unbounded, we define, for $w\in D\equiv[B(0),\infty)$, $\beta(w)\equiv\inf\{u:\bar{B}(u)=w\}$.

Theorem 2.3. *If* $B(u)$ *is unbounded,* $S\equiv\left\{w\in D:\beta(w)-\frac{a^2}{8}w>0\right\}$ *is not empty, and* u_0 *satisfies the condition*

$$(2.5)\qquad |u_0|^{(0)}_{(I)}<M\equiv\sup_{w\in S}\left(\beta(w)-\frac{a^2}{8}w\right)\leqq\infty,$$

then there exists a unique solution for (2.1)–(2.2) *belonging to* $H^{2+\alpha}(\infty,I)$. *Moreover, we have*

$$(2.6)\qquad |u|^{(0)}_{T(I)}\leqq|u_0|^{(0)}_{(I)}+\frac{a^2}{8}\inf\left\{w\in S:|u_0|^{(0)}_{(I)}<\beta(w)-\frac{a^2}{8}w\leqq M\right\}.$$

Proof. Take optionally $T\in(0,\infty)$ and $k\in S$ such that $|u_0|^{(0)}_{(I)}<\beta(k)-\frac{a^2}{8}k\leqq M$. Let $u(x,t)$ satisfy (2.1)–(2.2). Then, $V(x,t)\equiv u(x,t)+\frac{k}{2}\left(x-\frac{a}{2}\right)^2$ satisfies the equation (2.4) as K is replaced by k. Moreover, we have $B(u)-K<0$, therefore $|V|^{(0)}_{T(I)}$ and $|u|^{(0)}_{T(I)}$ are not larger than $|u_0|^{(0)}_{(I)}+\frac{a^2}{8}\cdot k$. This assertion results from the relations $B(u)\leqq\bar{B}(u)$ and $\bar{B}\left(|u_0|^{(0)}_{(I)}+\frac{a^2}{8}k\right)<\bar{B}(\beta(k))=k$, and from the use of a simple method of *reductio ad absurdum* and of the comparison theorem. Considering the way of having taken k, we obtain (2.6). Q.E.D.

2.2. Examples. We consider only the case of $I\equiv[0,1]$.

(i) For $\varphi(u)=1+u^2$ and $\psi(u)=u^2$, $B(u)=u^2/(1+u^2)$. $K\equiv\sup B(u)=1$. This example corresponds with Theorem 2.2.

(ii) For $\varphi(u)=1+u$ and $\psi(u)=u^2$, $B(u)=u^2/(1+u)$ is in a strict sense monotonically increasing and unbounded. In this example, which is related with Theorem 2.3, $S=(0,\infty)$. Thus, $M=\infty$. For the problem (2.1)–(2.2), the solution $u(x,t)$ does not blow up, and we have $|u|^{(0)}_{T(I)}\leqq|u_0|^{(0)}_{(I)}+C(|u_0|^{(0)}_{(I)})$.

(iii) For $\varphi(u)=1$ and $\psi(u)=u^2$, $S=(0,64)$ and $M=2(B(u)=u^2)$. Therefore, if $|u_0|<2$, then $u(x,t)$ does not blow up and it holds that $0\leqq u(x,t)<4$.

2.3. We give a more concrete result in

Theorem 2.4. (i) *If $B(u)\leqq C_1u^\gamma+C_2$ $(C_1, C_2\geqq 0;\ 0\leqq\gamma<1)$, then there exists a unique solution $u(x,t)\in H^{2+\alpha}(\infty, I)$ for* (2.1)–(2.2).

(ii) *If $B(u)>C_3u^\gamma-C_4$ $(C_3>0,\ C_4\geqq 0,\ \gamma>1)$, then the solution $u(x,t)$ blows up under some conditions on u_0. There are also cases to which Theorem* 2.3 *is applicable.*

(iii) *If $C_5u+C'\geqq B(u)\geqq C_5u-C''$ $(C_5>0;\ C', C''\geqq 0)$, then, under the conditions that $a_1\equiv C_5-(\pi/a)^2>0$, that*

$$(2.7)\qquad \begin{cases} C_0[u_0]\equiv\displaystyle\int_I \Phi(u_0(x))\cdot s(x)\,dx>\Phi\left(\frac{C''}{a_1}\right) \\ \left(\Phi(u)\equiv\displaystyle\int_0^u \frac{1}{\varphi(u)}du,\ s(x)\equiv\frac{\pi}{2a}\sin\frac{\pi x}{a}\right), \end{cases}$$

and that $W(Q)\equiv\displaystyle\int_{C_0}^{Q}\{\Phi^{-1}(Q)\}^{-1}dQ\,(\Phi(\infty)>Q\geqq C_0>\Phi(0)=0)$ is bounded, the solution $u(x,t)$ blows up. There are also cases to which Theorem 2.3 *is applicable.*

Proof. (i) $\exists k_0$ such that $(k_0,\infty)\subset S$.

(ii) In case that $C_0[u]$ is sufficiently large, $u(x,t)$ blows up. Because $J(t)\equiv\displaystyle\int_I u(x,t)\cdot s(x)dx$ satisfies an inequality

$$(2.8)\qquad \frac{1}{\varphi(0)}J(t)\geqq C_0[u_0]+\int_0^t\left\{C_3J(\tau)^\gamma-\left(\frac{\pi}{a}\right)^2J(\tau)-C_4\right\}d\tau.$$

For the latter part, e.g., see 2.2 (iii).

(iii) For the former part, it suffices to see that, by Jensen's inequality, it holds that

$$(2.9)\qquad \Phi(J(t))\geqq C_0[u_0]+\int_0^t\{a_1J(\tau)-C''\}d\tau.$$

For the latter part, e.g., see 2.2 (ii). Q.E.D.

3. We consider the following Cauchy problem:

$$(3.1)\qquad \begin{cases} u_t=\varphi(u)u_{xx}+\psi(u)\cdot h(x),\ (-\infty<x<\infty,\ t\geqq 0, \\ \qquad\qquad h(x)(\geqq 0)\in L_1(R_1)\cap H^\alpha(0<\alpha<1)), \\ u(x,0)=u_0(x)\ (\geqq 0)\in H^{2+\alpha}. \end{cases}$$

Theorem 3.1. *If $B(u)$ is bounded, then there exists a unique solution $u(x,t)$ for* (3.1) *belonging to $H^{2+\alpha}(\infty, R^1)$.*

Proof. Replace $\psi(u)$ in (3.1) by $K\cdot\varphi(u)(K\equiv\sup B(u))$. Apply the comparison theorem to (3.1) and the new equation. Thereafter, make use of procedures analogous to those in [5].

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