23. *Some Remarks on Zero Cycles on Abelian Varieties*

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Let A be an abelian variety of dimension n with identity element o over an algebraically closed field k of any characteristic, and let $CH_0(A)$ be the Chow group of 0-dimensional cycles on A. For any point $a \in A$, (a) denotes the 0-cycle of degree 1 defined by a. In this note we shall prove the following

Theorem 1. *A point $a \in A$ is torsion if and only if the 0-cycle $(a)-(o)$ is torsion of the same order in $CH_0(A)$.*

§1. **Preliminaries.** In this section we review some results developed in [1], which are needed in our proof. Let X be a smooth projective variety over an algebraically closed field k. For an integer $r \geqq 0$, the Chow group of codimension r, $CH^r(X)$, is defined to be the group of cycles on X of codimension r modulo rational equivalence. Put $CH^*(X)=\oplus_{r\geqq 0} CH^r(X)$. Then $CH^*(X)$ has a structure of graded ring with respect to intersection product ([2]), and is called the Chow ring of X. We also define the Chow group of dimension r, $CH_r(X)=CH^{n-r}(X)$ where $n=\dim X$, and put $CH_*(X)=\oplus_{r\geqq 0} CH_r(X)$. When $X=A$ is an abelian variety, $CH_*(A)$ also has a structure of graded ring with respect to Pontryagin product, defined as follows; let $\mu: A\times A \to A$ denote the group law of A. For two cycles $\sigma \in CH_s(A)$, $\tau \in CH_t(A)$, the Pontryagin product $\sigma * \tau$ is defined by $\sigma*\tau=\mu_*(\sigma\times\tau) \in CH_{s+t}(A)$. The group of 0-cycles $CH_0(A)\subset CH_*(A)$ forms a subring, and the subgroup of the 0-cycles of degree zero $I\subset CH_0(A)$ is an ideal of $CH_0(A)$. I is generated by cycles $(a)-(o)$, for $a\in A$. We have three lemmas by [1] and [3].

Lemma 1. *The sequence*

$$0 \longrightarrow I^{*2} \hookrightarrow I \xrightarrow{\varphi} A \longrightarrow 0$$

is exact, where $\varphi((a)-(o))=a$ for $a \in A$.

Lemma 2. *$I^{*(n+1)}=0$, where $I^{*(n+1)}$ denotes the $(n+1)$ times Pontryagin product of I, and $n=\dim A$.*

Lemma 3. *I is a divisible group.*

For the proofs, see [1].

§2. **Proof of the theorem.** Suppose $m((a)-(o))=0$ in $CH_0(A)$, then Lemma 1 implies $ma=o$ in A. Conversely, suppose $ma=o$ in A. For an integer i, define $f_i \in End(I)$ by the following formula;

$$f_i(x)=(((i+1)a)-(ia))*x \qquad (x\in I).$$

Trivially we obtain the following equalities:

$$(ia)*f_j=f_{i+j}. \tag{1}$$

$$\sum_{i=0}^{m-1} f_i=0. \tag{2}$$

Define $r_j \in End(I)$ $(j\geqq 0)$ by $r_j=((a)-(o))^{*j}*f_0$ $(j\geqq 1)$ and $r_0=f_0$. By binomial expansion, r_j can be written as a linear combination of f_i's;

$$r_j=\sum_{i=0}^{m-1} C(j,i)f_i, \qquad \text{where } C(j,i)\in \boldsymbol{Z}, \tag{3}$$

and

$$\sum_{i=0}^{m-1} C(j,i)=0 \qquad \text{if } j\geqq 1. \tag{4}$$

Lemma 4. $\sum_{i=0}^{m-1} (ia)*r_j=0$ $(j\geqq 0)$.

Proof. If $j=0$, the lemma is immediate from (2). If $j\geqq 1$,

$$\sum_{i=0}^{m-1} (ia)*r_j=\sum_{i=0}^{m-1}\left(\sum_{k=0}^{m-1} C(j,k)f_{i+k}\right)=\sum_{h=0}^{m-1}\left(\sum_{i=0}^{m-1} C(j,[h-i])f_h\right)=0,$$

where $[z]$ denotes the least non-negative integer which is congruent to z modulo m. This proves the lemma.

Lemma 5. $f_0=0 \in End(I)$.

Proof. By Lemma 2, we have $r_{n+1}=0$. Suppose $r_{k+1}=0$ $(k\geqq 0)$, then $0=r_{k+1}=(a)*r_k-r_k$, so we have $(ia)*r_k=r_k$ for any $i\in \boldsymbol{Z}$ and $0=\sum_{i=0}^{m-1}(ia)*r_k=mr_k$. Since $End(I)$ is torsion free by Lemma 3, $r_k=0$. By descending induction, we have $f_0=r_0=0$, which proves the lemma.

From this lemma, $f_0((a)-(o))=0$, so $(a)-(o)=(a)*((a)-(o))$ $=(ia)*((a)-(o))$ for $i\in \boldsymbol{Z}$. Then we have

$$m((a)-(o))=\sum_{i=0}^{m-1}(ia)*((a)-(o))=0.$$

This completes the proof of theorem.

Corollary. *Assume that the ground field k is the algebraic closure of a finite field. Then* $I^{*2}=0$.

Proof. In this case I is torsion by Theorem 1. This and Lemma 3 imply $I\otimes I=0$, so $I^{*2}=0$.

Remark. This corollary was stated in [1] (p. 223) as a result due to Swan. The proof given there also uses the fact that I is torsion, but the idea of deducing this fact is slightly different. Our Theorem gives perhaps a more natural approach to this corollary.

§3. Application. Let A be an abelian variety and suppose that the characteristic of the ground field is not equal 2. For arbitrary 2-division point $a\in A$, let $\sigma \in Aut(A)$ denote the translation by a. Then σ commutes with inverse operation of A, so it induces an automorphism $\tilde{\sigma}$ of $Y=Km(A)$, where $Km(A)$ denotes the desingularized Kummer variety associated to A.

Theorem 2. *The automorphism* $\tilde{\sigma}$ *acts trivially on the Chow group* $CH_0(Y)$, i.e. $\tilde{\sigma}_*=id$.

Proof. Let $I(A)$ (resp. $I(Y)$) denote the group of 0-cycles of degree 0 on A (resp. Y), and $p: A \to Y$ be the natural rational map. Using Chow's moving lemma ([2]), we can define $p_*: CH_0(A) \to CH_0(Y)$. If $y \in I(Y)$ then there exists $x \in I(A)$ such that $p_*(x) = y$. From Theorem 1, $\sigma_*(x) - x = f_0(x) = 0$, and so

$$\tilde{\sigma}_*(y) = \tilde{\sigma}_*(p_*(x)) = p_*(\sigma_*(x)) = p_*(x) = y.$$

On the other hand, if we denote by b a 4-division point such that $2b = a$, then $\deg(p_*((b))) = 1$ and $\tilde{\sigma}_*(p_*((b))) = p_*(\sigma_*((b))) = p_*((a+b)) = p_*((b))$. Hence $\tilde{\sigma}_*(y) = y$ for all $y \in CH_0(Y)$.

Remark and acknowledgements. In [4] H. Inose and M. Mizukami proved $I(X) = 0$ for certain surfaces of general type. Concerning this work T. Shioda remarked that the same method proves Theorem 2 for the Kummer surface associated to the Jacobian of a curve of genus 2. Our present work has been motivated by these results. We would like to express our deep application to Prof. T. Shioda, Mr. H. Inose and Mr. Mizukami.

References

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