79. Asymptotic Behavior of Iterates of Nonexpansive Mappings in Banach Spaces. II

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1. Introduction. Let $X$ be a Banach space and let $X^*$ be the dual space of $X$. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $(x, x^*)$. The duality mapping $F$ (multi-valued) from $X$ into $X^*$ is defined by

$$F(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 \text{ and } \|x^*\| = \|x\|\} \quad \text{for } x \in X.$$  

We say that $X$ is smooth, if $\lim_{t \to 0} t^{-1}(\|x+ty\| - \|x\|)$ exists for every $x$ and $y$ with $\|x\| = \|y\| = 1$. $F$ is single-valued if and only if $X$ is smooth. The duality mapping $F$ of a smooth Banach space $X$ is said to be weakly continuous at 0 if $\omega^{-1}(x_n) = 0$ in $X$ implies that $\{F(x_n)\}$ converges weakly* to 0 in $X^*$, where $\omega^{-1}(x_n)$ denotes the weak limit of $\{x_n\}$. It is easy to see that Hilbert space and $(l^p), 1 < p < \infty$, have this property.

Throughout the rest of this paper it is assumed that $X$ is a smooth and uniformly convex real Banach space having the duality mapping $F$ which is weakly continuous at 0, and $C$ is a nonempty closed convex subset of $X$. By $\text{Cont}(C)$ we mean that $T$ is a nonexpansive mapping from $C$ into itself, i.e., $T : C \to C$ satisfies $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed-points of $T$ will be denoted by $\text{Fix}(T)$.

In [5], Z. Opial proved the following: Let $T \in \text{Cont}(C)$ and $x \in C$. If $\|T^n - T^{n+1}\| = 0$, then the sequence $\{T^n x\}$ is weakly convergent to an element of $\text{Fix}(T)$. (Let $F_\mu$ be a duality mapping of $X$ into $X^*$ with gauge function $\mu$ (see [5]). We note here that $F_\mu$ is weakly continuous at 0 if and only if $F$ is weakly continuous at 0.)

The purpose of this note is to prove the following

Theorem. Let $T \in \text{Cont}(C)$ and $x \in C$. Then $\omega^{-1}(x)$ exists if and only if $\omega^{-1}(x)$ exists. Moreover $\omega^{-1}(x) \in \text{Fix}(T)$ if the weak limit exists.

In the case that $X$ is a Hilbert space, the theorem has been obtained by R. E. Bruck [2].

2. Proof of Theorem. In the preceding paper [4] the author proved the following: Let $T \in \text{Cont}(C)$ and $x \in C$. Then $\omega^{-1}(x)$ exists if and only if $\omega^{-1}(x)$ exists. The set of weak subsequential limits of $\{T^n x\}$. Therefore to prove Theorem it suffices to show the following
Proposition. Let $T \in \text{Cont}(C)$ and $x \in C$. If
\begin{equation}
(1) \text{w-lim}_{n \to \infty} (T^{n+1}x - T^n x) = 0,
\end{equation}
then $\omega_u(x) \subseteq \mathcal{F}(T)$.

Recall that $X$ is called uniformly convex if the modulus of convexity
\[ \delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \epsilon \right\} \]
is positive for every $\epsilon$ with $0 < \epsilon \leq 2$. Let $\alpha > 0$. It is easily seen that for every $\epsilon$ with $0 < \epsilon \leq 2\alpha$
\begin{equation}
(2) \|x\| \leq \alpha, \|y\| \leq \alpha \text{ and } \|x - y\| \geq \epsilon \text{ imply } \|x + y\|/2 \leq \alpha(1 - \delta(\epsilon/\alpha)).
\end{equation}

Let $\{x_n\}$ be a bounded sequence in $C$. Then there exists a unique point $c \in C$ such that
\[ \limsup_{n \to \infty} \|x_n - c\| < \limsup_{n \to \infty} \|x_n - x\| \quad \text{for } x \in C \setminus \{c\}. \]
(See [3].) The point $c$ is called the asymptotic center of $\{x_n\}$ with respect to $C$. By the weak continuity of $F$ at $0$ we have the following (see [4, Lemma (b)]):
\begin{equation}
(3) \text{Let } \{x\} \text{ be a sequence in } C. \text{ If } \text{w-lim}_{n \to \infty} x_n \text{ exists, then the weak limit is the asymptotic center of } \{x_n\} \text{ with respect to } C.
\end{equation}

Proof of Proposition. Let $u \in \omega_u(x)$. Then there is a subsequence $\{n_k\}$ of $\{n\}$ such that $w\text{-lim}_{k \to \infty} T^{n_k} x = u$. By (1) we have
\[ w\text{-lim}_{k \to \infty} T^{n_k + m} x = u \quad \text{for every nonnegative integer } m. \]
It follows from (3) that for every $m \geq 0$, $u$ is the asymptotic center of $\{T^{n_k + m} x ; k = 1, 2, \ldots\}$ with respect to $C$. Consequently
\[ \limsup_{k \to \infty} \|T^{n_k + m} x - u\| \leq \limsup_{k \to \infty} \|T^{n_k + m + 1} x - x\| \quad \text{for } z \in C \text{ and } m = 0, 1, \ldots. \]

Put
\[ r_m = \limsup_{k \to \infty} \|T^{n_k + m} x - u\| \quad \text{for } m = 0, 1, 2, \ldots. \]
Then by (4) and $T \in \text{Cont}(C)$ we have
\[ r_{m+1} = \limsup_{k \to \infty} \|T^{n_k + m + 1} x - u\| \leq \limsup_{k \to \infty} \|T^{n_k + m} x - u\| = r_m \quad \text{for } m = 0, 1, 2, \ldots. \]
Therefore $\{r_m\}$ is convergent to $r = \inf \{r_m : m \geq 0\}$.

We now prove that $u$ is a fixed-point of $T$. First, let $r = 0$. Since
\[ |(u - Tu, x^*)| \leq 2 \|x^*\| \|T^{n_k + m} x - u\| + \|(T^{n_k + m} x - T^{n_k + m + 1} x, x^*)| \quad \text{for } x^* \in X^*, \]
it follows from (1) that
\[ |(u - Tu, x^*)| \leq 2 \|x^*\| \limsup_{k \to \infty} \|T^{n_k + m} x - u\| = 2 \|x^*\| r_m \]
for every $x^* \in X^*$ and $m \geq 0$. By $\lim_{m \to \infty} r_m = r = 0$ we have
\[ (u - Tu, x^*) = 0 \quad \text{for every } x^* \in X^*, \text{ i.e., } Tu = u. \]

Next, let $r > 0$. We use the same argument as in the proof of Theorem in [1]. To prove $u \in \mathcal{F}(T)$ it suffices to show that $\|T^p u - u\| \to 0$ as $p \to \infty$. Suppose, for contradiction, that the sequence $\{\|T^p u - u\|\}$ does not converge to 0. Then there is a $d > 0$ and a subsequence $\{p_j\}$ of $\{p\}$ such that $r \geq d$ and $\|T^{p_j} u - u\| \geq d$ for all $j \geq 1$. We can choose an $\epsilon_0 > 0$ such that $(r + \epsilon_0)|1 - \delta(d/(r + \epsilon_0))| < r$. By $r = \lim_{m \to \infty} r_m$ there
exists a positive integer \( m_0 \) such that

\[
\limsup_{k \to \infty} \| T^{n+m} x - u \| = r_m < r + \varepsilon_0 \quad \text{for } m \geq m_0.
\]

Therefore for every \( m \geq m_0 \) there exists a positive integer \( k(m) \) such that

\[
(5) \quad \| T^{n+k} x - u \| < r + \varepsilon_0 \quad \text{for every } k \geq k(m).
\]

Take an integer \( j > 0 \) with \( p_j \geq m_0 \). By (5) we have that

\[
\| T^{p_j} u - T^{n+p_j} x \| \leq \| u - T^{n+p_j} x \| < r + \varepsilon_0 \quad \text{for } k \geq k(p_j)
\]

and

\[
\| u - T^{n+p_j} x \| < r + \varepsilon_0 \quad \text{for } k \geq k(2p_j).
\]

Since \( \| (T^{p_j} u - T^{n+p_j} x) - (u - T^{n+p_j} x) \| = \| T^{p_j} u - u \| \geq d \), it follows from

(2) that

\[
\| T^{n+p_j} x - (u + T^{p_j} u)/2 \| = \| (T^{p_j} u - T^{n+p_j} x) + (u - T^{n+p_j} x) \|/2 \leq (r + \varepsilon_0[1 - \delta(d/(r + \varepsilon_0))] \quad \text{for } k \geq \max\{k(p_j), k(2p_j)\},
\]

and hence

\[
\limsup_{k \to \infty} \| T^{n+p_j} x - (u + T^{p_j} u)/2 \| \leq (r + \varepsilon_0[1 - \delta(d/(r + \varepsilon_0))] < r.
\]

Since \( u \) is the asymptotic center of \( \{ T^{n+p_j} x; k = 1, 2, \ldots \} \) with respect to \( C \), we have

\[
r_{p_j} = \limsup_{k \to \infty} \| T^{n+p_j} x - u \| \leq \limsup_{k \to \infty} \| T^{n+p_j} x - (u + T^{p_j} u)/2 \| < r.
\]

This contradicts \( r = \inf \{ r_m : m \geq 0 \} \). Therefore \( \| T^{p_j} u - u \| \to 0 \) as \( p \to \infty \) and hence \( u \in \mathcal{F}(T) \).

Q.E.D.

3. An extension of Theorem. A mapping \( T: C \to C \) is called

asymptotically nonexpansive if there exists a sequence \( \{ a_n \} \) of positive numbers with \( \lim a_n = 1 \) such that

\[
\| T^n x - T^n y \| \leq a_n \| x - y \| \quad \text{for } x, y \in C \text{ and } n = 1, 2, \ldots.
\]

S. C. Bose [1] has extended Opial\'s theorem (which is stated in Introduction) to the case of asymptotically nonexpansive mapping. We can also extend our Theorem to the following form:

Theorem\'. Let \( T: C \to C \) be an asymptotically nonexpansive mapping and let \( x \in C \). Then \( \lim_{n \to \infty} T^n x \) exists if and only if \( \mathcal{F}(T) \neq \emptyset \) and \( \lim_{n \to \infty} (T^{n+1} x - T^n x) = 0 \). Moreover \( \lim_{n \to \infty} T^n x \in \mathcal{F}(T) \) if the weak limit exists.

Sketch of Proof. It suffices to prove the following (a) and (b):

(a) \( \lim_{n \to \infty} T^n x \) exists if and only if \( \mathcal{F}(T) \neq \emptyset \) and \( \omega_0(x) \subset \mathcal{F}(T) \);

(b) if \( \lim_{n \to \infty} (T^{n+1} x - T^n x) = 0 \) then \( \omega_0(x) \subset \mathcal{F}(T) \).

A proof of (a) may be found in [1]. To prove (b), let \( \lim_{n \to \infty} T^n x = u \) and put \( r_m = \limsup_{k \to \infty} \| T^{n+m} x - u \| \) for \( m \geq 0 \). Noting that \( u \) is the asymptotic center of \( \{ T^{n+m} x; k = 1, 2, \ldots \} \) with respect to \( C \) for every \( m \geq 0 \) and \( T \) is asymptotically nonexpansive, we have

\[
r_{m+1} \leq \limsup_{k \to \infty} \| T^{n+m+1} x - T^i u \| \leq a_{m} r_m \quad \text{for } m \geq 0 \text{ and } l \geq 0.
\]

It follows from \( \lim_{l \to \infty} a_l = 1 \) that \( \limsup_{l \to \infty} r_l = \limsup_{l \to \infty} r_{m+l} \leq r_m \) for
Thus \( \limsup_{t \to \infty} r_t \leq \liminf_{m \to \infty} r_m \), and therefore \( \{r_m\} \) is convergent. Put \( r = \lim_{m \to \infty} r_m \). Then, using the same argument as in the proof of Proposition, we obtain that \( u \in \mathcal{F}(T) \). (In this case, replace \( r_m < r + \varepsilon_0 \) for \( m \geq m_0 \)" in the proof of Proposition by \( r_m < r + \varepsilon_0/2 \) for \( m \geq m_0 \).) After this our argument is as follows. For every \( m \geq m_0 \) there is an integer \( k(m) > 0 \) such that \( \|T^{nk+m}x - u\| < r + \varepsilon_0/2 \) for \( k \geq k(m) \). Choose an integer \( j_0 > 0 \) such that \( p_j \geq m_0 \) and \( \alpha(p_j)(r + \varepsilon_0/2) < r + \varepsilon_0 \) for \( j \geq j_0 \). We have that \( \|T^{p_j}u - T^{nk+2p_j}x\| \leq \alpha(p_j)(r + \varepsilon_0/2) < r + \varepsilon_0 \) for \( k \geq k(p_j) \) and \( j \geq j_0 \), and \( \|u - T^{nk+2p_j}x\| < r + \varepsilon_0 \) for \( k \geq k(2p_j) \). These and \( \|T^{p_j}u - u\| \geq \delta \) yield \( \|T^{nk+2p_j}x - (u + T^{p_j}u)/2\| \leq (r + \varepsilon_0)[1 - \delta(d/(r + \varepsilon_0))] \) for \( k \geq \max\{k(p_j), k(2p_j)\} \) and \( j \geq j_0 \). Therefore \( r_{2p_j} = \limsup_{k \to \infty} \|T^{nk+2p_j}x - (u + T^{p_j}u)/2\| < (r + \varepsilon_0)[1 - \delta(d/(r + \varepsilon_0))] \) for \( j \geq j_0 \). This contradicts \( r = \lim_{m \to \infty} r_m \).

References