76. *A Difference Approach to Mikusiński's Operational Calculus*

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§ 1. **Introduction.** J. Mikusiński [1] introduced a clear and simple operational calculus to obtain explicitly the solution of Cauchy's problem for linear ordinary differential equations with constant coefficients. His calculus is based upon Titchmarsh's theorem concerning the convolution $\int_0^t f(t-s)g(s)ds$ of two continuous functions f and g defined on $[0, \infty)$. The present author should like to start with the fact that for the convolution ring of sequences the corresponding Titchmarsh-type theorem holds almost trivially. Hence we can easily introduce an operational calculus based upon this fact, and apply it to treat the Cauchy problem for linear difference-quotient equations with constant coefficients. By taking the limit of this treatment, we can prove the legitimacy of Mikusiński's operational method for differential equations without appealing to Titchmarsh's theorem.

§ 2. **An operational calculus.** We define by $\mathcal{S}$ the totality of complex number valued functions (sequences) f defined on the set $N\omega = \{\nu = j\omega\,;\, j=0,1,2,\cdots\}$ where ω is a non zero real number. In this paper, we write such functions by $\{f(\nu)\}$ or simply f; whereas $f(\nu)$ will mean the value at ν of the function f.

The convolution ring of sequences. For f and $g \in \mathcal{S}$, we define the sum $f+g$ by $\{f(\nu)+g(\nu)\}$ as usual. Clearly the set $\mathcal{S}$ becomes an additive group with respect to this addition. The zero element is the function which is identically equal to zero and that is written by 0. Next, we define the product (sequential convolution product) $f \cdot g$ of two functions f and g by the value at $\nu = j\omega$:

$$f \cdot g(\nu) = \omega \sum_{i=0}^{j} f((j-i)\omega)g(i\omega). \tag{1}$$

Proposition 1. *The set $\mathcal{S}$ is a commutative ring with respect to the addition $f+g$ and the multiplication $f \cdot g$.*
Moreover, the ring $\mathcal{S}$ has the following important property.

Proposition 2. *The ring $\mathcal{S}$ has no zero divisor, that is, $f \cdot g = 0$ implies that $f=0$ or $g=0$.*

Proof. Let $f \neq 0$. Then there exists the point ν_0 such that $f(\nu_0) \neq 0$ and $f(\nu)=0$ for all $\nu < \nu_0$. Therefore, by $f \cdot g(\nu_0) = f(\nu_0)g(0)$, we get $g(0)=0$. Now, suppose that $g(\nu)=0$ for all $\nu \leqq \mu$. Then, by $f \cdot g(\nu_0 + \mu$

$+\omega)=f(\nu_0)g(\mu+\omega)$, we must have $g(\mu+\omega)=0$. Thus we have proved that $g=0$.

The operator field. By the above proposition, we can construct the field $\hat{\mathcal{S}}$ of (sequential convolution) quotients in the following way. Introducing the quotient f/g of two functions f and $g \in \mathcal{S}$ with $g \neq 0$ and defining $f/g=h/k$ by $f \cdot k=g \cdot h$, we obtain as usual the field $\hat{\mathcal{S}}$ of quotients. Of course, the unit element in $\hat{\mathcal{S}}$ is g/g $(g \neq 0)$ and will be written simply as 1. We call the quotient f/g an operator so that our field $\hat{\mathcal{S}}$ shall be called the operator field as well. Every function $f \in \mathcal{S}$ can be considered as an operator since it can be identified with $f \cdot g/g$ $(g \neq 0)$.

The summation operator. We shall denote by h the operator defined by the function which is identically equal to 1, and call it the summation operator, since we get

$$h \cdot f(\nu)=\omega \sum_{i=0}^{j} f(i\omega) \tag{2}$$

where $f \in \mathcal{S}$ and $\nu=j\omega$. We often write $\sum_0^\nu f(\mu)\Delta\mu$ for the right-hand side. By this notation, we can rewrite (1) in the following way: $f \cdot g(\nu) =\sum_0^\nu f(\nu-\mu)g(\mu)\Delta\mu$.

Scalar operators. For any complex number α, the operator $[\alpha]$ defined by

$$[\alpha]=\{\alpha\}/h \tag{3}$$

is called a scalar operator, because as in Mikusiński [1] we have the following

Proposition 3. *For any $\alpha, \beta \in \boldsymbol{C}$ and $f, g \in \mathcal{S}$ with $g \neq 0$, the following formulas hold good:*

$$[\alpha]+[\beta]=[\alpha+\beta], \qquad [\alpha][\beta]=[\alpha\beta] \tag{4}$$

$$[\alpha] \cdot f=\alpha f \ (=\{\alpha f(\nu)\}), \qquad [\alpha] \cdot (f/g)=(\alpha f)/g \ (=\{\alpha f(\nu)/g(\nu)\}). \tag{5}$$

By (3) and (4), we can identify the operator $[\alpha]$ with the complex number α and by (5) we see that the effect of the operator $[\alpha]$ is exactly the α-times multiplication.

The difference-quotient operator. For any $f \in \mathcal{S}$, we define the difference Δf of the function f by $\Delta f(\nu)=f(\nu+\omega)-f(\nu)$ and the difference-quotient $(\Delta f)/(\Delta \nu)$ by

$$\Delta f(\nu)/\Delta\nu=(f(\nu+\omega)-f(\nu))/\omega. \tag{6}$$

Furthermore, we define the operator difference-quotient s by

$$s=1/h. \tag{7}$$

Remark. This operator s is not a scalar operator, since $1/h =\{1\}/\{1\}^2$.

Proposition 4. *For any $f \in \mathcal{S}$, the following formula holds good:*

$$\Delta f/\Delta\nu=f \cdot s-f(0)+(\omega\Delta f/\Delta\nu)\cdot s \tag{8}$$

($f(0)$ means the scalar operator $[f(0)]$).

Proof.

$$\left(\frac{\Delta f}{\Delta \nu}\right)\cdot h=\left\{\frac{\Delta f}{\Delta \nu}(\nu)\right\}\cdot h=\left\{\sum_0^{\nu}\frac{\Delta f}{\Delta \nu}(\mu)\Delta\mu\right\}=\{f(\nu+\omega)-f(0)\}$$
$$=\{f(\nu)\}-\{f(0)\}+\left\{\omega\frac{\Delta f}{\Delta \nu}(\nu)\right\}=f-[f(0)]\cdot h+\left(\omega\frac{\Delta f}{\Delta \nu}\right).$$

By multiplying the both sides by $s=1/h$, we have (8).

Corollary.

$$(9)\quad \frac{\Delta^k f}{\Delta \nu^k}=f\cdot s^n-f(0)\cdot s^{n-1}-\frac{\Delta f}{\Delta \nu}(0)\cdot s^{n-2}-\cdots-\frac{\Delta^{k-1} f}{\Delta \nu^{k-1}}(0)$$
$$+\left(\omega\frac{\Delta f}{\Delta \nu}\right)\cdot s^n+\left(\omega\frac{\Delta^2 f}{\Delta \nu^2}\right)\cdot s^{n-1}+\cdots+\left(\omega\frac{\Delta^k f}{\Delta \nu^k}\right)\cdot s$$

Proposition 5. *For any scalar $\alpha\in C$, we have*

$$(10)\quad 1/(s-\alpha)=\{((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+\omega)}\}\in\mathcal{S}.$$

Proof. Substitute the function $((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+\omega)}$ into (8). Then we have

$$(1-\omega\alpha)^{-1}\alpha\cdot\{((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+\omega)}\}$$
$$=(1-\omega\alpha)^{-1}s\cdot\{((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+\omega)}\}-(1-\omega\alpha)^{-1}.$$

By multiplying the both sides by $(1-\omega\alpha)$, we obtain

$$\alpha\{((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+\omega)}\}=s\cdot\{((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+\omega)}\}-1.$$

Corollary.

$$(11)\quad \frac{1}{(s-\alpha)^k}=\left\{\frac{(\nu+\omega)(\nu+2\omega)\cdots(\nu+(k-1)\omega)}{(k-1)!}((1-\omega\alpha)^{-1/\omega\alpha})^{\alpha(\nu+k\omega)}\right\}\in\mathcal{S}.$$

§3. Applications to differential equations. In this section, we shall consider the Cauchy problem for the following ordinary differential equation written in matrix form:

$$(12)\quad dy/dt=Ay+z,\qquad y(0)=\beta$$

where $A=(\alpha_{ij})$, $\beta=(\beta_j)$, $y=(y_j)$ and $z=(z_j)$; α_{ij} and β_j being constants and z_j continuous functions $(i, j=1, 2, \cdots, m)$.

Difference-quotient equations and their operational treatment. Consider the difference-quotient equation:

$$(13)\quad \Delta f/\Delta\nu=Af+g,\qquad f(0)=\beta$$

corresponding to differential equation (12). Here $f=(f_j)$ and $g=(g_j)$, g_j being the restriction of z_j to the set $N\omega$, i.e., $g_j(\nu)=z_j(\nu)$ for $\nu\in N\omega$ $(j=1, 2, \cdots, m)$. We shall apply our operational calculus to this equation. By Proposition 4, we have

$$(14)\quad \Delta f/\Delta\nu=s\cdot f-f(0)+s\cdot(\omega\Delta f/\Delta\nu)$$
$$(f(0)={}^t([f_1(0)], \cdots, [f_m(0)])).$$

Hence, by (13), we obtain the following operational equation:

$$(15)\quad (sI-A)\cdot f=\beta+g-s\cdot(\omega(\Delta f/\Delta\nu))\qquad (\beta={}^t([\beta_1], \cdots, [\beta_m]))$$

where I stands for the unit matrix. Therefore, we get

$$(16)\quad f=(sI-A)^{-1}\cdot\beta+(sI-A)^{-1}\cdot g-(sI-A)^{-1}\cdot s\cdot(\omega(\Delta f/\Delta\nu)).$$

$(sI-A)^{-1}$ means the inverse matrix of $(sI-A)$.

An approximate solution of equation (12). We put $\omega=p/n$, where p is a positive real number and n a positive integer. In order to make clear the dependence of functions f on n, we shall write ${}_nf$ by attaching the suffix n.

For ${}_nf\in\mathcal{S}$, we define the continuous function ${}_n\bar{f}$ by the polygonal curve which is given by connecting each vertex $(\nu, {}_nf(\nu))$ in succession by segment $(\nu\in N\omega,\ \omega=p/n)$. Also for a vector $F=({}_nf_j)$ with ${}_nf_j\in\mathcal{S}$, the vector-valued continuous function is defined by $\bar{M}=({}_n\bar{f}_j)$. We call ${}_n\bar{f}$ the polygonal extension of ${}_nf$ and $\bar{F}$ that of F.

Let ${}_nf$ be the solution of difference-quotient equation (13) which corresponds to $\omega=p/n$. Then we see that the graph of the polygonal extension of ${}_nf$ coinsides with a Cauchy polygon of differential equation (12). Therefore, we get:

Proposition 6. *The polygonal extension* ${}_n\bar{f}$ *of the solution* ${}_nf$ *of difference-quotient equation* (13) *is uniformly convergent to the solution* y *of differential equation* (12), *as* n *tends to the infinity. In particular, we have*

$$y(p)=\lim_{n\to\infty}{}_n\bar{f}(p)=\lim_{n\to\infty}{}_nf(p).$$

To obtain the concrete expression of this solution, we decompose the right-hand side of (16) into two parts f^* and f^{**}:

$$f^*=(sI-A)^{-1}\cdot\beta+(sI-A)^{-1}\cdot g, \tag{17}$$

$$f^{**}=(sI-A)^{-1}\cdot s\cdot(\omega(\Delta f/\Delta\nu)). \tag{18}$$

By the way, the ij-component a_{ij} of $(sI-A)^{-1}$ is a rational function of s and the degree of the denominator is greater than that of the numerator. Thus, a_{ij} is decomposed into partial fractions as follows: $a_{ij}=\sum_{q=1}^{m}\gamma_{ijq}/(s-\alpha_q)^{k_q}$. Therefore, we get

$$\text{the } i\text{-th component of } f^*=\sum_{j=1}^{m}\sum_{q=1}^{m}\gamma_{ijq}(s-\alpha_q)^{-k_q}\cdot(\beta_j+g_j). \tag{19}$$

Hence, by (11), each component of f^* is contained in $\mathcal{S}$. On the other hand, for each component a'_{ij} of $(sI-A)^{-1}\cdot s$, we have:

$$a'_{ij}=\gamma'_{ij}+\sum_{q=1}^{m}\gamma'_{ijq}/(s-\alpha_q)^{k_q}.$$

Therefore, we get

$$\text{the } i\text{-th component of } f^{**}=\frac{p}{n}\sum_{j=1}^{m}\left(\gamma'_{ij}\frac{\Delta f_j}{\Delta\nu}+\sum_q\gamma'_{ijq}(s-\alpha_q)^{-k_q}\cdot\frac{\Delta f_j}{\Delta\nu}\right) \tag{20}$$

and so, we see that each component of f^{**} also belongs to $\mathcal{S}$. Thus, again to make clear the dependence of these functions on n, we write ${}_nf^*$ and ${}_nf^{**}$ so that, by (16),

$${}_nf={}_nf^*-{}_nf^{**}. \tag{21}$$

Now, we are able to obtain explicitly the solution y of differential equation (12). Our procedure is divided into two steps.

Step 1. We shall prove

(22) $$y(p)=\lim_{n\to\infty} {}_nf^*(p).$$

Step 2. The right-hand side of (22) shall be given, by (17), explicitly in the following way.

We write $(\widetilde{sI-A})^{-1}$ for the matrix which is obtained by changing $1/(s-\alpha)^k$ in the components of the matrix $(sI-A)^{-1}$ by the continuous function $t^{k-1}e^{\alpha t}/(k-1)!$. Then we have, at point p,

(23) $$\lim_{n\to\infty} {}_nf^* = (\widetilde{sI-A})^{-1}\beta * + (\widetilde{sI-A})^{-1} * z,$$

where, putting $(c_{ij})=(\widetilde{sI-A})^{-1}$, $*$ is defined as follows: For $z=(z_j)$ $(j=1,\cdots,m)$,

$$(\widetilde{sI-A})^{-1}*z=(c_{ij})*(z_j)=\Big(\sum_{j=1}^m c_{ij}*z_j\Big)\Big(c_{ij}*z_j(t)=\int_0^t c_{ij}(t-s)z_j(s)ds\Big),$$

while for $\beta=(\beta_j)$ β_j being a numerical constant $(j=1,\cdots,m)$,

$$(\widetilde{sI-A})^{-1}*\beta=(c_{ij})*(\beta_j)=\Big(\sum_{j=1}^m c_{ij}\beta_j\Big)$$

$(c_{ij}\beta_j$: β_j-times multiplication of $c_{ij})$. Hence by (22), the solution y of (12) is given by the right-hand side of (23).

In this way, the legitimacy of Mikusiński's operational calculus for differential equations can be derived without appealing to Titchmarsh's theorem. To prove the above two steps, we shall prepare the following three lemmas.

Lemma 1. *Let ${}_nv$ and ${}_nw$ be elements of $\mathcal{S}$, and let v and w be continuous functions on $[0,p]$. If $\lim_{n\to\infty} {}_n\overline{v}=v$ and $\lim_{n\to\infty} {}_n\overline{w}=w$ uniformly on $[0,p]$. Then $\lim_{n\to\infty}({}_n\overline{v}{}_n\overline{w})=vw$ uniformly on $[0,p]$, where ${}_nv{}_nw$ means the pointwise product of ${}_nv$ and ${}_nw$, and vw that of v and w.*

Proof. This lemma will be proved by

$$|{}_n\overline{v_n w}(t)-{}_n\overline{v}{}_n\overline{w}(t)| \leqq \frac{1}{4}\max_{\nu=0,p/n,\cdots,(n-1)p/n}\Big|\Big({}_nv\Big(\nu+\frac{p}{n}\Big)-{}_nv(\nu)\Big)\Big({}_nw\Big(\nu+\frac{p}{n}\Big)-{}_nw(\nu)\Big)\Big|.$$

To prove this inequality, we put $f={}_n\overline{v_n w}-{}_n\overline{v}{}_n\overline{w}$. Then we see that f is a quadratic function of t $(\nu\leqq t\leqq \nu+p/n)$ with $f(\nu)=0$ and $f(\nu+p/n)=0$, so that

$$\max_{t\in[\nu,\nu+p/n]}|f(t)|=\Big|f\Big(\frac{\nu+(\nu+p/n)}{2}\Big)\Big|.$$

Therefore

$$|f(t)|\leqq \max_{\nu=0,p/n,\cdots,(n-1)p/n}\ \max_{t\in[\nu,\nu+p/n]}|f(t)| = \max_{\nu=0,p/n,\cdots,(n-1)p/n}\Big|f\Big(\frac{\nu+(\nu+p/n)}{2}\Big)\Big|.$$

Since ${}_n\overline{v}$, ${}_n\overline{w}$ and ${}_n\overline{v_n w}$ are linear functions of t $(\nu\leqq t\leqq \nu+p/n)$, we have

$$\Big|f\Big(\frac{\nu+(\nu+p/n)}{2}\Big)\Big| =4^{-1}|2({}_n\overline{v_n w}(\nu)+{}_n\overline{v_n w}(\nu+p/n))-({}_n\overline{v}(\nu)+{}_n\overline{v}(\nu+p/n))({}_n\overline{w}(\nu) +{}_n\overline{w}(\nu+p/n)|$$

$$=4^{-1}\,|2({}_nv_nw(\nu)+{}_nv_nw(\nu+p/n))-({}_nv(\nu)+{}_nv(\nu+p/n))\cdot({}_nw(\nu)+{}_nw(\nu+p/n))|$$

$$=4^{-1}\,|({}_nv(\nu+p/n)+{}_nv(\nu)({}_nw(\nu+p/n)-{}_nw(\nu))|.$$

As an application of Lemma 1, we give

Lemma 2. *Let ${}_n\bar{v}$ and ${}_n\bar{w}$ be polygonal extensions of ${}_nv$ and ${}_nw$ respectively, and let as $n\to\infty$ these polygonal extensions be uniformly convergent to continuous functions v and w on $[0,p]$ respectively. Then, we have*

$$\lim_{n\to\infty} {}_nv\cdot{}_nw(p)=v*w(p) \qquad \left(=\int_0^p v(p-s)w(s)ds\right).$$

We omit the proof.

Lemma 3. *Put*

$${}_nu_{\alpha,k}(\nu)=\frac{(\nu+p/n)(\nu+2p/n)\cdots(\nu+(k-1)p/n)}{(k-1)!}\left(\left(1-\frac{p\alpha}{n}\right)^{-n/p\alpha}\right)^{\alpha(\nu+k(p/n))}.$$

Then, ${}_n\bar{u}_{\alpha,k}(t)$ is uniformly convergent as $n\to\infty$ to $(t^{k-1}/(k-1)!)e^{\alpha t}$ on $[0,p]$.

Proof. ${}_nu(\nu)=((1+p\alpha/n)^{n/p\alpha})^{\alpha\nu}$ is the solution of the difference-quotient equation

$$\Delta_nu/\Delta\nu=\alpha_nu,\ {}_nu(0)=1 \qquad (\Delta\nu=\omega=p/n),$$

which corresponds to the differential equation

$$dy/dt=\alpha y \qquad y(0)=1.$$

Therefore, by Proposition 6, we see that $\lim_{n\to\infty}{}_n\bar{u}(t)=e^{\alpha t}$ on $[0,p]$. Now, we have, by taking $\nu=jp/n$,

$$|{}_nu_{\alpha,1}(\nu)-(1+p\alpha/n)_nu(\nu)|=|\{1-(1-(p\alpha/n)^2)^{j+1}\}/(1-p\alpha/n)^{j+1}|$$
$$\leqq\{(1+|p\alpha/n|^2)^{j+1}-1\}/(1-|p\alpha/n|)^{j+1}$$
$$\leqq\{(1+|p\alpha/n|^2)^{n+1}-1\}/(1-|p\alpha/n|)^{n+1}.$$

Thus we see $\lim_{n\to\infty}{}_n\bar{u}_{\alpha,1}(t)=\lim_{n\to\infty}(1+p\alpha/n)\cdot{}_n\bar{u}(t)=e^{\alpha t}$ uniformly on $[0,p]$, and so the case $k=1$ is proved. We omit the proof of the general case.

Proof of Step 1. We have $\lim_{n\to\infty}{}_n\bar{f}=y$ uniformly on $[0,p]$ by Proposition 6, and so, by $\Delta_nf/\Delta\nu=A_nf+g$, we obtain $\lim_{n\to\infty}\overline{\Delta_nf/\Delta\nu}=Ay+z$. Hence, by (20), Lemmas 2 and 3, we have $\lim_{n\to\infty}{}_nf^{**}(p)=0$. Thus, by (21), we obtain $\lim_{n\to\infty}{}_nf^*(p)=\lim_{n\to\infty}{}_nf(p)=y(p)$.

Proof of Step 2. Since z is continuous on $[0,p]$ and $g(\nu)=z(\nu)$ $(\nu=0,p/n,\cdots,(n-1)p/n,p)$, we have $\lim_{n\to\infty}{}_n\bar{g}=z$ uniformly on $[0,p]$. Hence, by (17), (19), Lemmas 2 and 3, we obtain $\lim_{n\to\infty}{}_nf^*=(sI-A)^{-1}*\beta+(sI-A)*z$.

Reference

[1] J. Mikusiński: Operational Calculus. Pergamon Press (1959).