70. Remarks on the Existence of Finite Invariant Measures for Groups of Measurable Transformations

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0. Introduction. Throughout this note let $(X, \mathfrak{B}, m)$ be a finite measure space and let G be an infinite group of invertible bi-measurable non-singular transformations of X onto itself. A measure μ on $(X, \mathfrak{B})$ is called *G-invariant* if $\mu(gE)=\mu(E)$ for all $g \in G$ and $E \in \mathfrak{B}$. By A. Hajian and Y. Ito [1] it is proved that there exists a finite G-invariant measure on $(X, \mathfrak{B})$ equivalent to m if and only if in $\mathfrak{B}$ there does not exist any weakly G-wandering set of positive m-measure. Making use of the elegant result, in this note, we shall give some necessary and sufficient conditions for the existence of a finite G-invariant measure on $(X, \mathfrak{B})$ equivalent to m. Our results have been shown by Hopf [3], Kubokawa [4], Hajian and Kakutani [2] for the case when G is a cyclic group.

1. The main theorem. To state our results, we begin with some definitions. By N we denote the set of all positive integers. In what follows let A, B, A_i, B_i $(i \in N)$ and W be subsets of X in $\mathfrak{B}$.

Definition 1. *A is equivalent to B under G, denoted by $A \sim B$, if A and B can be expressed as countable disjoint union $A=\bigcup_{i=1}^{\infty} A_i$ and $B=\bigcup_{i=1}^{\infty} B_i$ such that there exists a sequence $\{g_i ; i \in N\}$ in G satisfying $g_i A_i = B_i$ for all $i \in N$.*

Definition 2. *A is G-bounded if $m(A-B)=0$ for any $B \subset A$ with $B \sim A$.*

Definition 3. *$(X, \mathfrak{B}, m)$ is G-compact if for any $\varepsilon>0$ there corresponds a $\delta>0$ such that $m(A)<\delta$ and $B \sim A$ imply $m(B)<\varepsilon$.*

Definition 4. *W is weakly G-wandering if there exists a sequence $\{g_i ; i \in N\}$ in G such that $g_i W \cap g_j W=\emptyset$ for all $i, j \in N$ with $i \neq j$.*

Definition 5. *A family Λ of measures on $(X, \mathfrak{B})$ is equi-uniformly absolutely continuous with respect to m if for any $\varepsilon>0$ there corresponds a $\delta>0$ such that $m(B)<\delta$ implies $\lambda(B)<\varepsilon$ for all $\lambda \in \Lambda$.*

For every $g \in G$ let m_g be the measure on $(X, \mathfrak{B})$ defined by $m_g(E) = m(gE)$ for any $E \in \mathfrak{B}$. Now we consider the following conditions:

(0) *There exists a finite G-invariant measure μ on $(X, \mathfrak{B})$ which is equivalent to m.*

(1) *$(X, \mathfrak{B}, m)$ is G-compact.*

(2) *X is G-bounded.*

(3) *Every set A in $\mathfrak{B}$ is G-bounded.*

(4) *The family $\{m_g; g\in G\}$ of measures on $(X,\mathfrak{B})$ is equi-uniformly absolutely continuous with respect to m.*

(5) $\inf_{g\in G} m(gE)>0$ *for any $E\in\mathfrak{B}$ with $m(E)>0$.*

(6) *In $\mathfrak{B}$ there does not exist any weakly G-bounded set of positive m-measure.*

Then the next result is due to A. Hajian and Y. Ito [1].

Lemma 1. *The conditions* (0), (5) *and* (6) *are mutually equivalent.*

By virtue of Lemma 1, we have

Theorem 1. *The conditions* (0), (1), (2), (3) *and* (4) *are mutually equivalent.*

Proof. The implication (1)$\Rightarrow$(4)$\Rightarrow$(5) is seen easily from Definitions 3 and 5. So, according to Lemma 1, it suffices to prove the implication (0)$\Rightarrow$(1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(6). Let μ be a finite G-invariant measure on $(X,\mathfrak{B})$ equivalent to m. Since μ and m are mutually uniformly absolutely continuous to each other, for any $\varepsilon>0$ there corresponds a $\delta'>0$ such that $\mu(B)<\delta'$ implies $m(B)<\varepsilon$, and also for this δ' there corresponds a $\delta>0$ such that $m(A)<\delta$ implies $\mu(A)<\delta'$. So if $m(A)<\delta$ and $A\sim B$, then $\mu(A)=\mu(B)<\delta'$ and $m(B)<\varepsilon$. Therefore $(X,\mathfrak{B},m)$ is G-compact, and (0) implies (1). Let $Y_1\in\mathfrak{B}$ be a proper subset of X which is equivalent to X under G. Then we can find inductively a decreasing sequence $\{Y_i; i\in N\}$ in $\mathfrak{B}$ such that

$$(*)\qquad \begin{aligned} &X\sim Y_1\sim Y_2\sim\cdots\sim Y_n\sim Y_{n+1}\cdots \quad\text{and}\\ &X-Y_1\sim Y_1-Y_2\sim\cdots\sim Y_n-Y_{n+1}\sim\cdots. \end{aligned}$$

Suppose that $(X,\mathfrak{B},m)$ is G-compact. Then, as $\lim_{n\to\infty} m(Y_n-Y_{n+1})=0$ by $(*)$, it follows from Definition 3 that $m(X-Y_1)=0$. So X is G-bounded, and (1) implies (2). For any $A\in\mathfrak{B}$ let $B\in\mathfrak{B}$ satisfy $B\sim A$ and $B\subset A$. We put $C=A-B$, $D=X-A$ and $Y=C\cup D$. Then we see that $X\sim Y$. If X is G-bounded, then $m(C)=m(X-Y)=0$, and hence A is also G-bounded. So (3) follows from (2). Finally let $W\in\mathfrak{B}$ be weakly G-wandering and $\{g_i; i\in N\}$ be a sequence in G such that $g_iW\cap g_jW=\emptyset$ for all $i,j\in N$ with $i\neq j$. We set $A=\bigcup_{i=1}^{\infty} g_iW$ and $B=\bigcup_{i=2}^{\infty} g_iW$. Then clearly $B\sim A$ and $B\subset A$. Hence $m(g_1W)=m(A-B)=0$ and $m(W)=0$ if (3) holds. So (6) follows from (3). Consequently the theorem is proved.

2. Remarks. 1) The implication (0)$\Rightarrow$(2) is shown directly. In fact let μ be the measure as in (0) and $Y\in\mathfrak{B}$ satisfy $X\sim Y$. Then, as $\mu(X)=\mu(Y)$, we have $\mu(X-Y)=0$ and hence $m(X-Y)=0$. Further the conditions (0), (2) and (3) are mutually equivalent also for the case when $(X,\mathfrak{B},m)$ is a σ-finite measure space, because Lemma 1 and the implication (0)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(6) are valid also for this case.

2) When G is a cyclic group, the equivalency (0)$\Leftrightarrow$(1), (0)$\Leftrightarrow$(2) and

(0)$\Leftrightarrow$(4) are proved in [4, Theorem 1], [3, Theorem 4] and [2, Theorem 1] respectively.

References

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