4. Asymptotic Properties of Solutions of n-th Order Differential Equations with Deviating Argument

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1. Introduction. This paper is concerned with the asymptotic behavior of solutions of differential equations of the form

(A) $$Lx(t)+f(t, x(g(t)))=0,$$

where the differential operator L is defined by

$$Lx(t)=(p_{n-1}(t)(p_{n-2}(t)(\cdots(p_1(t)x'(t))'\cdots)')')'.$$

L may also be written in the form $L=L_n$, where the operators L_i are recursively defined by

$$L_1x(t)=x'(t),\quad L_ix(t)=(p_{i-1}(t)L_{i-1}x(t))',\quad 2\leqq i\leqq n.$$

The conditions we always assume for p_i, g, f are as follows:

(a) Each $p_i(t)$ is continuous and positive on $[a, \infty)$ and

$$\int_a^\infty \frac{dt}{p_i(t)}=\infty,\qquad 1\leqq i\leqq n-1;$$

(b) $g(t)$ is continuous on $[a, \infty)$ and $\lim_{t\to\infty} g(t)=\infty$;

(c) $f(t, x)$ is continuous on $[a, \infty)\times(-\infty, \infty)$ and $|f(t, x)|\leqq\omega(t, |x|)$ for $(t, x)\in[a, \infty)\times(-\infty, \infty)$, where $\omega(t, r)$ is continuous on $[a, \infty)\times[0, \infty)$ and nondecreasing in r.

Equation (A) is called *superlinear* or *sublinear* according to whether $\omega(t, r)/r$ is nondecreasing or nonincreasing in r for $r>0$.

A function $x(t)$ defined on some half-axis $[T_x, \infty)$ is said to be a solution of (A) if $L_1x(t), L_2x(t), \cdots, L_nx(t)$ exist and are continuous on (T, ∞), where $T>T_x$ is such that $g(t)>T_x$ for $t>T$, and if $x(t)$ satisfies (A) on (T, ∞). Hereafter our attention will be restricted to solutions of (A) which are nontrivial on any infinite subintervals of $[T_x, \infty)$. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros; otherwise the solution is said to be *nonoscillatory*.

The asymptotic properties of second order functional differential equations with a general deviating argument have recently been studied by Kitamura and Kusano [2]. The object of this paper is to extend the theory developed in [2] to higher-order equations of the form (A). Of particular interest is an analysis of the effect that $g(t)$ can have on the growth or decay of solutions of equation (A) which is either superlinear or sublinear. The results are stated without proofs; an exposition in full detail will appear elsewhere.

2. Possible behavior of all solutions. We use the following notation throughout the paper:

$$\phi_{i,i}(t,s)=1 \qquad (0\leqq i\leqq n-1),$$

$$\phi_{i,j}(t,s)=\int_s^t \frac{ds_{i+1}}{p_{i+1}(s_{i+1})}\int_s^{s_{i+1}} \frac{ds_{i+2}}{p_{i+2}(s_{i+2})}\int_s^{s_{i+2}}\cdots\int_s^{s_{j-1}} \frac{ds_j}{p_j(s_j)} \qquad (0\leqq i<j\leqq n-1),$$

$$P_i(t)=\phi_{0,i}(t,a),\quad Q_i(t)=|\phi_{i,n-1}(a,t)| \quad (0\leqq i\leqq n-1),$$

$$A_i(t)=\frac{P_i(t)}{P_{i-1}(t)},\quad B_i(t)=\max\left\{1,\frac{P_{i-1}(t)Q_{i-1}(t)}{P_i(t)Q_i(t)}\right\} \quad (1\leqq i\leqq n-1),$$

$$g^*(t)=\max\{g(t),t\},\qquad g_*(t)=\min\{g(t),t\},$$

$$h^*(t)=\sup_{a\leqq s\leqq t} g^*(s),\qquad h_*(t)=\inf_{s\geqq t} g_*(s).$$

The basic result of this paper is the following theorem which describes the possible behavior of all solutions of (A).

Theorem 1. *Suppose that either* (A) *is superlinear and*

$$(1)\qquad \int^\infty \frac{A_i(g^*(t))}{A_i(g(t))}B_i(t)Q_i(t)\omega(t,cP_i(g(t)))dt<\infty$$

for all $c>0$ *and* i *with* $1\leqq i\leqq n-1$, *or* (A) *is sublinear and*

$$(2)\qquad \int^\infty A_i(g^*(t))B_i(t)Q_i(t)\omega(t,cP_{i-1}(g(t)))dt<\infty$$

for all $c>0$ *and* i *with* $1\leqq i\leqq n-1$.

If $x(t)$ *is a solution of* (A), *then one of the following cases holds*:

(I) $\displaystyle\limsup_{t\to\infty}\frac{|x(t)|}{P_{n-1}(t)}=\infty$;

(II) *There exist an integer* k, $0\leqq k\leqq n-1$, *and a nonzero number* c_k *such that*

$$\lim_{t\to\infty}\frac{x(t)}{P_k(t)}=c_k;$$

(III) $\displaystyle\lim_{t\to\infty} x(t)=0$.

In the proof of this theorem the following lemma is crucial.

Lemma 1. *Let* $1\leqq k\leqq n-1$ *and suppose that*

$$\int^\infty \frac{A_k(g^*(t))}{A_k(g(t))}B_k(t)Q_k(t)\omega(t,P_k(g(t)))dt<\infty$$

if (A) *is superlinear and that*

$$\int^\infty A_k(g^*(t))B_k(t)Q_k(t)\omega(t,P_{k-1}(g(t)))dt<\infty$$

if (A) *is sublinear.*

If $x(t)$ *is a solution of* (A) *such that* $x(t)=o(P_k(t))$ *as* $t\to\infty$, *then* $x(t)=O(P_{k-1}(t))$ *as* $t\to\infty$.

We say that condition (G^*) [resp. (G_*)] is satisfied if there is a sequence $\{t_\nu\}_{\nu=1}^\infty$ such that $t_\nu\to\infty$ as $\nu\to\infty$ and $h^*(t_\nu)=t_\nu$ [resp. $h_*(t_\nu)=t_\nu$] for $\nu=1,2,\cdots$.

These additional conditions on $g(t)$ are used to exclude Case (I) or (III) from the possibilities listed in Theorem 1.

Lemma 2. *Suppose that* (A) *is superlinear and* (G_*) *is satisfied. If*

$$\int^{\infty} Q_0(t)\omega(t, 1)dt < \infty,$$

then every solution $x(t)$ *of* (A) *satisfies* $\limsup_{t\to\infty} |x(t)| > 0$.

Lemma 3. *Suppose that* (A) *is sublinear and* (G^*) *is satisfied. If*

$$\int^{\infty} \omega(t, P_{n-1}(g(t)))dt < \infty,$$

then every solution $x(t)$ *of* (A) *satisfies* $x(t) = O(P_{n-1}(t))$ *as* $t \to \infty$.

3. Behavior of nonoscillatory solutions. A fundamental set of solutions of the equation $Lx(t)=0$ is given by $\{P_k(t) : 0 \leqq k \leqq n-1\}$. It is not difficult to find a sufficient condition for (A) to have solutions which are asymptotic to a $P_k(t)$ as $t \to \infty$.

Theorem 2. *Let* $0 \leqq k \leqq n-1$ *and suppose that*

$$(3) \qquad \int^{\infty} Q_k(t)\omega(t, cP_k(g(t)))dt < \infty \qquad \textit{for some } c > 0.$$

Then (A) *has solutions* $x(t)$, $y(t)$ *such that*

$$\lim_{t\to\infty} \frac{x(t)}{P_k(t)} = \frac{c}{2}, \qquad \lim_{t\to\infty} \frac{y(t)}{P_k(t)} = -\frac{c}{2}.$$

This theorem extends a recent result due to Granata [1]. The proof is standard; we transform (A) into the integral equation

$$u(t) = \alpha P_k(t) + \Phi_k u(t),$$

where $\alpha = c/2$ or $-c/2$, and

$$\Phi_k u(t) = \int_t^{\infty} \phi_{0,n-1}(t, s) f(s, u(g(s)))ds \qquad \text{if } k=0,$$

$$\Phi_k u(t) = \int_T^t \frac{ds_1}{p_1(s_1)} \int_T^{s_1} \cdots \int_T^{s_{k-1}} \frac{ds_k}{p_k(s_k)} \int_{s_k}^{\infty} \phi_{k,n-1}(s_k, s) f(s, u(g(s)))ds \qquad \text{if } 1 \leqq k \leqq n-1,$$

T being sufficiently large, and then solve it with the help of the Schauder-Tychonoff fixed point theorem.

Suppose that (A) is either superlinear or sublinear. Then the hypotheses of Theorem 1 guarantee that (3) holds for all $c > 0$ and each k, $0 \leqq k \leqq n-1$. It follows therefore that under the hypotheses of Theorem 1 equation (A) actually possesses nonoscillatory solutions which are asymptotic to $P_k(t)$ as $t \to \infty$ for every k, $0 \leqq k \leqq n-1$. Suppose moreover that $xf(t, x)$ is of constant sign. In this case, if $x(t)$ is a nonoscillatory solution of (A), then each of $L_1x(t), L_2x(t), \cdots, L_{n-1}x(t)$ is eventually of constant sign. By analyzing this situation carefully and using condition (G^*) or (G_*) if necessary, we can show that in certain cases all nonoscillatory solutions of (A) are subject to Case (II) of Theorem 1, that is, they behave like the solutions of $Lx(t)=0$.

Theorem 3. *Let n be even and $xf(t,x) \geqq 0$ on $[a, \infty) \times (-\infty, \infty)$. Suppose that the hypotheses of Theorem* 1 *are satisfied.*

Then, any nonoscillatory solution of (A) *behaves as in Case* (II) *of Theorem* 1.

Theorem 4. *Let n be odd. Suppose that*

(i) (A) *is superlinear, $xf(t,x) \geqq 0$ on $[a, \infty) \times (-\infty, \infty)$, (G_*) is satisfied and* (1) *holds for all $c>0$ and every i, $1 \leqq i \leqq n-1$, or*

(ii) (A) *is sublinear, $xf(t,x) \leqq 0$ on $[a, \infty) \times (-\infty, \infty)$, (G^*) is satisfied and* (2) *holds for all $c>0$ and every i, $1 \leqq i \leqq n-1$.*

Then, any nonoscillatory solution of (A) *behaves as in Case* (II) *of Theorem* 1.

4. Behavior of oscillatory solutions. The asymptotic behavior of oscillatory solutions of (A) is described in the following theorem.

Theorem 5. (i) *Assume that* (A) *is superlinear and (G_*) is satisfied. If* (1) *holds for all $c>0$ and every i, $1 \leqq i \leqq n-1$, then every oscillatory solution $x(t)$ of* (A) *has the property*

$$\limsup_{t \to \infty} \frac{|x(t)|}{P_{n-1}(t)} = \infty.$$

(ii) *Assume that* (A) *is sublinear and (G^*) is satisfied. If* (2) *holds for all $c>0$ and every i, $1 \leqq i \leqq n-1$, then every oscillatory solution of* (A) *has the property*

$$\lim_{t \to \infty} x(t) = 0.$$

As a consequence of Theorem 5 we have the following nonoscillation result for *almost linear* equations of the form (A).

Theorem 6. *Suppose that*

$$|f(t,x)| \leqq q(t)\,|x| \qquad \text{for } (t,x) \in [a, \infty) \times (-\infty, \infty),$$

where $q(t)$ is continuous and positive on $[a, \infty)$. Suppose in addition that both (G^*) and (G_*) are satisfied. If

$$\int^{\infty} A_i(g^*(t)) B_i(t) P_{i-1}(g(t)) Q_i(t) q(t) dt < \infty \qquad \text{for } 1 \leqq i \leqq n-1,$$

then all solutions of (A) *are nonoscillatory.*

References

[1] A. Granata: Singular Cauchy problems and asymptotic behaviour for a class of n-th order differential equations. Funkcial. Ekvac. (to appear).

[2] Y. Kitamura and T. Kusano: Asymptotic properties of solutions of two-dimensional differential systems with deviating argument. Hiroshima Math. J. (to appear).