On vanishing theorems for analytic spaces

By Osamu FUJINO

Department of Mathematics, Graduate School of Science, Kyoto University,
Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

(Communicated by Shigefumi MORI, M.J.A., March 12, 2024)

Abstract: This is an announcement of our new vanishing theorems for projective morphisms between complex analytic spaces. We established a complex analytic generalization of Kollár's torsion-freeness and vanishing theorem for analytic simple normal crossing pairs. Although our results may look artificial, they have already played a crucial role for the study of minimal models in the complex analytic setting.

Key words: Vanishing theorems; complex analytic spaces; mixed Hodge modules; minimal model program.

1. Introduction. This is a short announcement of our new vanishing theorems for projective morphisms between complex analytic spaces. All the details will be published in [F6].

In his monumental paper [K], Kollár generalized the Kodaira vanishing theorem for complex projective varieties. His results consist of injectivity, torsion-free, and vanishing theorems. We have already had a powerful generalization of Kollár's package for *reducible* algebraic varieties (for the details, see, for example, [F4, Chapter 5]), which plays a crucial role for the study of log canonical pairs, semi-log canonical pairs, and quasi-log schemes in the theory of minimal models of algebraic varieties (see [F1], [F3], [F4, Chapter 6], and so on). Hence it was highly desirable to establish an analytic generalization (see [F4, Remark 5.8.3] and [F5, 1.10]). Roughly speaking, from the Hodge theoretic viewpoint, Kollár's original result in [K] is *pure* and the generalization in [F4, Chapter 5] is *mixed*. Recently, in [F6], we established an appropriate generalization of [F4, Chapter 5] for projective morphisms of complex analytic spaces. By this new generalization, we can translate the results in [F1], [F3], and [F4, Chapter 6] into the ones for projective morphisms between complex analytic spaces (see [F7] and [F8]). More precisely, in [F7], we proved the cone and contraction theorem of normal pairs for projective morphisms between complex analytic spaces as an application of [F6]. Then, in [F8], we discussed quasi-log structures for complex analytic spaces. We have already established the theory of minimal models for projective morphisms of complex analytic spaces with mild singularities in [F5], which is an analytic generalization of the great work of Birkar–Cascini–Hacon–M^{c}Kernan. We note that [F5] does not need our new vanishing theorems. The Kawamata–Viehweg vanishing theorem for projective morphisms of complex analytic spaces is sufficient for [F5]. Finally, we recommend the reader who is interested in vanishing theorems and the minimal model program to see [F4, Chapter 3].

In this paper, every complex analytic space is assumed to be *Hausdorff* and *second-countable*. We will freely use the standard notation in [F1], [F4], [F5], and so on. Let us prepare various definitions in order to explain our new vanishing theorems.

1.1 (Analytic globally embedded simple normal crossing pairs)**.** Let X be a simple normal crossing divisor on a smooth complex analytic space M and let B be an $\mathbf{R}$-divisor on M such that the support of $B+X$ is a simple normal crossing divisor on M and that B and X have no common irreducible components. Then we put $D := B|_X$ and consider the pair (X, D). The pair (X, D) is called an *analytic globally embedded simple normal crossing pair*.

Analytic globally embedded simple normal crossing pairs naturally appear when we use the resolution of singularities.

2020 Mathematics Subject Classification. Primary 32L20; Secondary 14E30.

doi: 10.3792/pjaa.100.005

1.2 (Analytic simple normal crossing pairs). If the pair (X, D) is locally isomorphic to an analytic globally embedded simple normal crossing pair and the irreducible components of X and D are all smooth, then (X, D) is called an *analytic simple normal crossing pair*. When (X, D) is an analytic simple normal crossing pair, X has an invertible dualizing sheaf ω_X.

1.3 (Strata). Let (X, D) be an analytic simple normal crossing pair. Let $\nu\colon X^\nu \to X$ be the normalization. Let Θ be the union of $\nu_*^{-1}D$ and the inverse image of the singular locus of X. If W is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ), then W is called a *stratum* of (X, D). Note that X^ν is smooth and the support of Θ is a simple normal crossing divisor on X^ν. We also note that if the coefficients of D are in $[0, 1]$ then (X^ν, Θ) is log canonical.

In the theory of minimal models, the notion of $\mathbf{R}$-line bundles is indispensable.

1.4 ($\mathbf{R}$-line bundles and $\mathbf{Q}$-line bundles). Let X be a complex analytic space and let $\mathrm{Pic}(X)$ be the group of line bundles on X, that is, the *Picard group* of X. An element of $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ (resp. $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$) is called an $\mathbf{R}$*-line bundle* (resp. a $\mathbf{Q}$*-line bundle*) on X. As usual, in this paper, we write the group law of $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ additively for simplicity of notation.

We need Siu's theorem to state our result.

1.5 (Associated subvarieties, see [Si]). Let $\mathcal{F}$ be a coherent sheaf on a complex analytic space X. Then there exists a locally finite family $\{Y_i\}_{i\in I}$ of complex analytic subvarieties of X such that

$$\mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = \{\mathfrak{p}_{x,1}, \dots, \mathfrak{p}_{x,r(x)}\}$$

holds for every point $x \in X$, where $\mathfrak{p}_{x,1}, \dots, \mathfrak{p}_{x,r(x)}$ are the prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $Y_{i,x}$ of Y_i at x with $x \in Y_i$. We note that each Y_i is called an *associated subvariety* of $\mathcal{F}$.

The following theorem is the main result of [F6], which is obviously an analytic generalization of [F4, Theorem 5.6.2].

Theorem 1.6 (Main theorem, [F6, Theorem 1.1]). *Let (X, Δ) be an analytic simple normal crossing pair such that the coefficients of Δ are in $[0, 1]$. Let $f\colon X \to Y$ be a projective morphism to a complex analytic space Y and let $\mathcal{L}$ be a line bundle on X. Let q be an arbitrary nonnegative integer. Then we have the following properties.*

(i) (*Strict support condition*). *If $\mathcal{L} - (\omega_X + \Delta)$ is f-semiample, then every associated subvariety of $R^q f_*\mathcal{L}$ is the f-image of some stratum of (X, Δ).*

(ii) (*Vanishing theorem*). *If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbf{R}} f^*\mathcal{H}$ holds for some π-ample $\mathbf{R}$-line bundle $\mathcal{H}$ on Y, where $\pi\colon Y \to Z$ is a projective morphism to a complex analytic space Z, then we have $R^p\pi_*R^q f_*\mathcal{L} = 0$ for every $p > 0$.*

Since we treat complex analytic spaces, we have to be careful about some basic definitions.

1.7 (Globally $\mathbf{R}$-Cartier divisors). In Theorem 1.6, we always implicitly assume that Δ is *globally* $\mathbf{R}$*-Cartier*, that is, Δ is a finite $\mathbf{R}$-linear combination of Cartier divisors on X. We note that if the number of the irreducible components of the support of Δ is finite then Δ is globally $\mathbf{R}$-Cartier. This condition is harmless to applications because the restriction of Δ to a relatively compact open subset of X has only finitely many irreducible components in its support. Under the assumption that Δ is globally $\mathbf{R}$-Cartier, we can obtain an $\mathbf{R}$-line bundle $\mathcal{N}$ naturally associated to $\mathcal{L} - (\omega_X + \Delta)$, which is a hybrid of line bundles $\mathcal{L}$ and ω_X and a globally $\mathbf{R}$-Cartier divisor Δ. The assumption in Theorem 1.6 (i) means that $\mathcal{N}$ is a finite positive $\mathbf{R}$-linear combination of f-semiample line bundles on X. The assumption in Theorem 1.6 (ii) means that $\mathcal{N} = f^*\mathcal{H}$ holds in $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$.

1.8. Let $X = \mathbf{C}$ and let $\{P_n\}_{n=1}^\infty$ be a set of mutually distinct discrete points of X. Then $\Delta = \sum_{n=1}^\infty \frac{1}{n} P_n$ is a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X. However, it is not a finite $\mathbf{R}$-linear combination of Cartier divisors on X. Hence it is not a globally $\mathbf{R}$-Cartier divisor.

Theorem 1.6 (ii) can be generalized as follows. It is an analytic generalization of [F4, Theorem 5.7.3].

Theorem 1.9 (Vanishing theorem of Reid–Fukuda type, [F6, Theorem 1.2]). *Let (X, Δ) be an analytic simple normal crossing pair such that the coefficients of Δ are in $[0, 1]$. Let $f\colon X \to Y$ and $\pi\colon Y \to Z$ be projective morphisms between complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbf{R}} f^*\mathcal{H}$ holds such that $\mathcal{H}$ is an $\mathbf{R}$-line bundle, which is nef and log big over Z with respect to $f\colon (X, \Delta) \to Y$, then $R^p\pi_*R^q f_*\mathcal{L} = 0$ holds for every $p > 0$ and every q.*

The definition of nef and log big line bundles in Theorem 1.9 is as follows:

1.10 (Nef and log big line bundles)**.** In Theorem 1.9, we note that $\mathcal{H}$ is said to be *nef and log big over* Z *with respect to* $f\colon (X, \Delta) \to Y$ if $\mathcal{H} \cdot C \geq 0$ holds for every projective integral curve C on Y such that $\pi(C)$ is a point and $\mathcal{H}|_{f(W)}$ can be written as a finite positive **R**-linear combination of π-big line bundles on $f(W)$ for every stratum W of (X, Δ).

It is more or less well known that Theorem 1.6 follows from Theorem 1.11 (i) and (ii) below and that Theorem 1.11 (iii) is an easy consequence of Theorem 1.11 (i) and (ii). Hence all we have to do is to establish Theorem 1.11 (i) and (ii). We will prove them in Section 2.

Theorem 1.11 (Kollár's package for analytic simple normal crossing pairs)**.** *Let* (X, D) *be an analytic simple normal crossing pair such that* D *is reduced and let* $f\colon X \to Y$ *be a projective morphism of complex analytic spaces. Then we have the following properties.*

(i) (*Strict support condition*). *Every associated subvariety of* $R^q f_* \omega_X(D)$ *is the* f*-image of some stratum of* (X, D) *for every* q.

(ii) (*Vanishing theorem*). *Let* $\pi\colon Y \to Z$ *be a projective morphism between complex analytic spaces and let* $\mathcal{A}$ *be a* π*-ample line bundle on* Y*. Then*

$$R^p \pi_* (\mathcal{A} \otimes R^q f_* \omega_X(D)) = 0$$

holds for every $p > 0$ *and every* q.

(iii) (*Injectivity theorem*). *Let* $\mathcal{L}$ *be an* f*-semiample line bundle on* X*. Let* s *be a nonzero element of* $H^0(X, \mathcal{L}^{\otimes k})$ *for some nonnegative integer* k *such that the zero locus of* s *does not contain any strata of* (X, D)*. Then, for every* q*, the map*

$$\begin{aligned} \times s\colon R^q f_*(\omega_X(D) \otimes \mathcal{L}^{\otimes l}) \\ \to R^q f_*(\omega_X(D) \otimes \mathcal{L}^{\otimes k+l}) \end{aligned}$$

induced by $\otimes s$ *is injective for every positive integer* l.

Kollár's original result in [K] is a special case of Theorem 1.11.

1.12 (Kollár's original statement)**.** If X is a smooth projective variety with $D = 0$ and $f\colon X \to Y$ is a projective surjective morphism onto a projective variety Y in Theorem 1.11 (i), then the strict support condition is nothing but Kollár's torsion-freeness of $R^q f_* \omega_X$ (see [K, Theorem 2.1 (i)]). We further assume that Z is a point in Theorem 1.11 (ii). Then we can recover Kollár's vanishing theorem (see [K, Theorem 2.1 (iii)]). If X is a smooth projective variety, $D = 0$, and Y is a point, then Theorem 1.11 (iii) coincides with Kollár's original injectivity theorem (see [K, Theorem 2.2]). Hence Theorem 1.11 generalizes Kollár's original statement in [K].

Our approach to Theorem 1.11 in [F6], which is completely different from the argument in [F4, Chapter 5], is very simple. By using a spectral sequence coming from Saito's theory of mixed Hodge modules (see Theorem 2.3 below), we can reduce Theorem 1.11 to a well-known simpler case due to Takegoshi (see Theorem 2.1 below). Hence our proof of Theorem 1.11 in [F6] uses the semisimplicity of polarizable Hodge modules. The advantage of this approach is to clarify the meaning of the strict support condition in Theorem 1.11 (i). We note that the above semisimplicity comes from the semisimplicity of polarizable variations of pure Hodge structure since polarizable Hodge modules are uniquely determined by their generic variations of pure Hodge structure. We note that the reader can find an alternative approach to Theorem 1.11, which is free from Saito's theory of mixed Hodge modules and only depends on the semisimplicity of polarizable variations of pure Hodge structure, in [FF].

2. Sketch of Proof.

In this section, we will briefly discuss how to prove the theorems in Section 1. As we have already explained in Section 1, we reduce the problem to a well-known simpler case due to Takegoshi by using a spectral sequence coming from the theory of mixed Hodge modules.

The following theorem is a special case of Takegoshi's result (see [T]). It is a complex analytic generalization of Kollár's torsion-freeness and vanishing theorem.

Theorem 2.1 (see [T])**.** *Let* $f\colon X \to Y$ *be a projective surjective morphism from a smooth irreducible complex analytic space* X*. Then* $R^q f_* \omega_X$ *is a torsion-free coherent sheaf on* Y *for every* q.

Furthermore, let $\pi\colon Y \to Z$ *be a projective morphism between complex analytic spaces and let* $\mathcal{A}$ *be a* π*-ample line bundle on* Y*. Then* $R^p \pi_*(\mathcal{A} \otimes R^q f_* \omega_X) = 0$ *holds for every* $p > 0$ *and every* q.

Although Takegoshi's complex analytic approach to Kollár's theorems in [T] is interesting, we do not discuss it here since the statement of

Theorem 2.1 is sufficient for our purposes in this paper. For an alternative approach to Theorem 2.1, we recommend the reader to see [F2, Corollaries 1.2 and 1.5].

2.2. Let (X, D) be an analytic simple normal crossing pair such that D is reduced. For any positive integer k, we put

$$X^{[k]} := \{x \in X \mid \mathrm{mult}_x X \geq k\}^{\sim},$$

where $Z^{\sim}$ denotes the normalization of Z. Then $X^{[k]}$ is the disjoint union of the intersections of k irreducible components of X, and is smooth. We have a reduced simple normal crossing divisor $D^{[k]} \subset X^{[k]}$ defined by the pull-back of D by the natural morphism $X^{[k]} \to X$. For any nonnegative integer l, we put

$$D^{[k,l]} := \{x \in X^{[k]} \mid \mathrm{mult}_x D^{[k]} \geq l\}^{\sim}.$$

We note that $D^{[k,0]} = X^{[k]}$ holds by definition. We also note that $\dim D^{[k,l]} = n + 1 - k - l$, where $n = \dim X$. In this situation, W is a stratum of (X, D) if and only if W is the image of an irreducible component of $D^{[k,l]}$ for some $k > 0$ and $l \geq 0$.

One of the main ingredients of [F6] is the following result coming from Saito's theory of mixed Hodge modules (see [Sa1] and [Sa2]).

Theorem 2.3 ([FFS, Corollary 1 and 4.7. Remark]). *Let (X, D) be an analytic simple normal crossing pair with $\dim X = n$ such that D is reduced and let $f\colon X \to Y$ be a projective morphism to a smooth complex analytic space Y. Then there is the weight spectral sequence*

$$ {}_F E_1^{-q, i+q} = \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D^{[k,l]}/Y} \Rightarrow R^i f_* \omega_{X/Y}(D),$$

degenerating at E_2, and its E_1-differential d_1 splits so that the ${}_F E_2^{-q,i+q}$ are direct factors of ${}_F E_1^{-q,i+q}$.

For the details of the weight spectral sequence of mixed Hodge modules on Y necessary for Theorem 2.3, see [FFS]. Once we know Theorems 2.1 and 2.3, it is easy to prove Theorem 1.11. Here, we only prove Theorem 1.11 (i) and (ii).

Proof of Theorem 1.11. First we prove (i). Since the problem is local, we may assume that Y is a closed analytic subspace of a polydisc Δ^m. By replacing Y with Δ^m, we may further assume that Y itself is a polydisc. In this case, we can use Theorem 2.3. We note that $\omega_Y \simeq \mathcal{O}_Y$ holds. By Theorem 2.1,

$$ {}_F E_1^{-q,i+q} \simeq \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D^{[k,l]}}$$

satisfies the strict support condition, that is, every associated subvariety of

$$ {}_F E_1^{-q,i+q} \simeq \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D^{[k,l]}}$$

is the f-image of some stratum of (X, D). By Theorem 2.3, the associated subvariety of ${}_F E_2^{-q,i+q} = {}_F E_\infty^{-q,i+q}$ is the f-image of some stratum of (X, D). This implies that $R^q f_* \omega_X(D)$ satisfies the desired strict support condition. Next, we treat (ii). We may assume that Z is a polydisc and Y is a closed analytic subspace of $Z \times \mathbf{P}^n$. By applying Theorem 2.3 to $f\colon X \to Y \hookrightarrow Z \times \mathbf{P}^n$, we obtain the following spectral sequence

$$E_1^{-q,i+q} = \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D^{[k,l]}} \Rightarrow R^i f_* \omega_X(D)$$

which degenerates at E_2 such that its E_1-differential d_1 splits. By Theorem 2.1, we obtain

$$R^p \pi_* (\mathcal{A} \otimes E_1^{-q,i+q}) = 0$$

for every $p > 0$. Since the $E_2^{-q,i+q} = E_\infty^{-q,i+q}$ are direct factors of $E_1^{-q,i+q}$, we have

$$R^p \pi_* (\mathcal{A} \otimes E_2^{-q,i+q}) = 0$$

for every $p > 0$. This implies that

$$R^p \pi_* (\mathcal{A} \otimes R^q f_* \omega_X(D)) = 0$$

holds for every $p > 0$. This is what we wanted. □

Since the proof of Theorem 1.6 in [F6] is somewhat technical, we only give a sketch here. For the details, see [F6, Sections 4 and 5].

Sketch of Proof of Theorem 1.6. For (i), we take an arbitrary point $y \in Y$. We may shrink Y around y suitably. By perturbing Δ, we may further assume that Δ is a $\mathbf{Q}$-divisor. Then we reduce the problem to the case where (X, Δ) is an analytic globally embedded simple normal crossing pair (see [F6, Lemma 5.1]). By repeatedly taking suitable finite covers, we can reduce the problem to Theorem 1.11 (i). For (ii), we can shrink Z suitably and may assume that Δ is a $\mathbf{Q}$-divisor. As for (i), we make (X, Δ) an analytic globally embedded simple normal crossing pair (see [F6, Lemma 5.1]) and repeatedly take suitable finite covers. Then we see that (ii) follows from Theorem 1.11 (ii). □

Theorem 1.9 follows from Theorem 1.6.

Sketch of Proof of Theorem 1.9. We need no new ideas for the proof of Theorem 1.9. The proof of [F4, Theorem 5.7.3] can work with some suitable modifications. Theorem 1.9 can be seen as a corollary of Theorem 1.6. □

2.4 (Traditional approach versus new approach)**.** Here we briefly review the main difference between the approach adopted in [F4, Chapter 5] and the one explained in this paper. Note that in [F4, Chapter 5] everything is assumed to be *algebraic*. In [F4, Chapter 5], we first establish an injectivity theorem by using the theory of mixed Hodge structures on cohomology with compact support (see [F4, Sections 5.4 and 5.5]). By this Hodge theoretic injectivity theorem, we can check that the injectivity theorem like Theorem 1.11 (iii) holds true in the algebraic setting (see, for example, [F4, Theorem 5.6.1]). It is well known that we can quickly recover Theorem 1.11 (i) and (ii) once we obtain Theorem 1.11 (iii). When X is algebraic, we can always take a compactification $\overline{X}$ of X and use the mixed Hodge structures on $\overline{X}$. On the other hand, if X is not algebraic, then we can not always compactify X. Thus, the argument in [F4, Chapter 5] does not work when X is not algebraic. Anyway, a key result in the approach of [F4, Chapter 5] is the injectivity theorem coming from the theory of mixed Hodge structures on cohomology with compact support. In the approach explained in this paper, injectivity theorems do not play an important role.

Acknowledgements. The author thanks Professor Taro Fujisawa very much. He was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H00974, JP21H04994.

References

[F1] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[F2] O. Fujino, A transcendental approach to Kollár's injectivity theorem II, J. Reine Angew. Math. **681** (2013), 149–174.

[F3] O. Fujino, Fundamental theorems for semi log canonical pairs, Algebr. Geom. **1** (2014), no. 2, 194–228.

[F4] O. Fujino, *Foundations of the minimal model program*, MSJ Mem., **35** Mathematical Society of Japan, Tokyo, 2017.

[F5] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, arXiv:2201.11315.

[F6] O. Fujino, Vanishing theorems for projective morphisms between complex analytic spaces, to appear in Math. Res. Lett.

[F7] O. Fujino, Cone and contraction theorem for projective morphisms between complex analytic spaces, arXiv:2209.06382.

[F8] O. Fujino, On quasi-log structures for complex analytic spaces, arXiv:2209.11401.

[FF] O. Fujino and T. Fujisawa, Variation of mixed Hodge structure and its applications, arXiv: 2304.00672.

[FFS] O. Fujino, T. Fujisawa and M. Saito, Some remarks on the semipositivity theorems, Publ. Res. Inst. Math. Sci. **50** (2014), no. 1, 85–112.

[K] J. Kollár, Higher direct images of dualizing sheaves. I, Ann. of Math. **123** (1986), no. 1, 11–42.

[Sa1] M. Saito, Modules de Hodge polarisables, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995.

[Sa2] M. Saito, Mixed Hodge modules, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.

[Si] Y. T. Siu, Noether-Lasker decomposition of coherent analytic subsheaves, Trans. Amer. Math. Soc. **135** (1969), 375–385.

[T] K. Takegoshi, Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms, Math. Ann. **303** (1995), no. 3, 389–416.