Legendre magnetic flows for geodesic spheres in a complex projective space

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Abstract: On a geodesic sphere in a complex projective space, we have Sasakian magnetic fields induced by the almost contact metric structure. In this paper, we investigate their magnetic flows on the unit sphere subbundle of the bundle of the contact distribution over this geodesic sphere, and show that they are smoothly conjugate to each other.

Key words: Sasakian magnetic fields; Legendre curves; magnetic flows; conjugate; geodesic spheres.

1. Introduction. On each geodesic sphere in a complex projective space, we have an almost contact metric structure induced by complex structure on the ambient space. By use of the canonical closed 2-form induced by this structure, we can define magnetic fields on this geodesic sphere, which are called Sasakian magnetic fields or contact magnetic fields (cf. [3,6,7]). Just like geodesics give the geodesic flow, we obtain magnetic flows on the unit tangent bundle from trajectories of charged particles under the influence of magnetic fields. In this paper, we investigate the relationship between two magnetic flows for a given geodesic sphere in a complex projective space.

In his papers [1,2], the second author studied trajectories for Kähler magnetic fields which are induced by complex structure on a complex projective space $\mathbf{C}P^n$ and on a complex hyperbolic space $\mathbf{C}H^n$. He showed that magnetic flows on the unit tangent bundle of $\mathbf{C}P^n$ are smoothly conjugate to each other, and that those on the unit tangent bundle of $\mathbf{C}H^n$ are classified into three conjugate classes. Since some geodesic spheres in $\mathbf{C}P^n$ are manifolds which are so called Sasakian space forms, manifolds having constant ϕ-sectional curvatures, it is natural to consider that we can find a corresponding property on magnetic flows for these manifolds.

Being different from trajectories for Kähler magnetic fields, strengths acting on trajectories for Sasakian magnetic fields depend on trajectories. This makes our treatment of these trajectories a bit complicated. Though Maeda and the second author studied all geodesics on Sasakian space forms in [4], the authors restricted themselves to trajectories which are orthogonal to characteristic vector fields and investigate their length spectrum in [10]. In this paper, following this line, we investigate restricted magnetic flows obtained by trajectories orthogonal to characteristic vector fields. We take horizontal lifts of these trajectories through a Hopf fibration. Considering the expression of the horizontal lift of the contact distribution, we describe the flow on a subbundle of the tangent bundle of complex Euclidean space. We then show that restricted magnetic flows associated with Sasakian magnetic fields on each geodesic sphere in $\mathbf{C}P^n$ are conjugate to each other.

2. Trajectories for Sasakian magnetic fields. A real hypersurface M in a Kähler manifold $\widetilde{M}$ with complex structure J admits an almost contact metric structure $(\xi, \phi, \eta, \langle\,,\rangle)$ (c.f. [5]). With a unit normal (local) vector field $\mathcal{N}$ of M in $\widetilde{M}$, the characteristic vector field ξ is defined by $\xi = -J\mathcal{N}$, the 1-form η is given by $\eta(v) = \langle v, \xi\rangle$, the structure tensor field ϕ is defined by $\phi(v) = Jv - \eta(v)\mathcal{N}$, and $\langle\,,\rangle$ is the metric induced from the one on $\widetilde{M}$. On this real hypersurface M, we have a natural closed 2-form $\mathbf{F}_\phi$ defined by $\mathbf{F}_\phi(u,v) = \langle u, \phi v\rangle$ for $u, v \in T_pM$ at an arbitrary point $p \in M$. Its constant multiple $\mathbf{F}_\kappa = \kappa\mathbf{F}_\phi$ ($\kappa \in \mathbf{R}$) is said to be a *Sasakian magnetic field* or *contact magnetic field* (for magnetic fields, see [11]). A smooth curve γ parameterized by its arc-length is said to be a *trajectory* for $\mathbf{F}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. Clearly, when $\kappa = 0$, it is a geodesic. For this

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trajectory γ, we set $\rho_\gamma(t) = \eta(\dot\gamma(t))$ and call this function its *structure torsion*. Since we have $\|\nabla_{\dot\gamma}\dot\gamma\| = |\kappa|\sqrt{1-\rho_\gamma^2}$, it measures the influence of the magnetic field. In order to calculate the differential of the structure torsion, we denote by $\widetilde\nabla$ and ∇ the covariant differentiations on $\widetilde M$ and M, respectively. We have the Gauss and Weingarten formulas

$$\widetilde\nabla_X Y = \nabla_X Y + \langle AX, Y\rangle\mathcal{N}, \quad \widetilde\nabla_X\mathcal{N} = -A\mathcal{N}$$

for vector fields X, Y tangent to M with the shape operator A with respect to $\mathcal{N}$. These lead us to $\nabla_X\xi = \phi AX$. We therefore find that

$$\begin{aligned}\frac{d}{dt}\rho_\gamma &= \dot\gamma\langle\dot\gamma,\xi\rangle = \langle\kappa\phi\dot\gamma,\xi\rangle + \langle\dot\gamma,\phi A\dot\gamma\rangle\\ &= \langle\dot\gamma,\phi A\dot\gamma\rangle = -\langle A\phi\dot\gamma,\dot\gamma\rangle,\end{aligned}$$

hence have

$$\frac{d}{dt}\rho_\gamma = \frac{1}{2}\langle\dot\gamma,(\phi A - A\phi)\dot\gamma\rangle. \tag{2.1}$$

Thus, the structure torsion for a trajectory γ is not constant along γ, in general.

In this paper we study magnetic flows for geodesic spheres in a complex projective space. Just like geodesics induce the geodesic flow, trajectories for $\mathbf{F}_\kappa$ induce *magnetic flow* $\mathbf{F}_\kappa\varphi_t : UM \to UM$ on the unit tangent bundle UM of M. That is, for $v \in UM$ we denote by γ_u the trajectory of initial vector v, and set $\mathbf{F}_\kappa\varphi_t(v) = \dot\gamma_u(t)$. Let $G(r)$ denote a geodesic sphere of radius r in a complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c. It has two principal curvatures. Denoting its shape operator with respect to the inner unit normal $\mathcal{N}$ by A, we have

$$\begin{aligned}A\xi &= \sqrt{c}\cot\sqrt{c}r\,\xi,\\ Av &= \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r\,v \quad (v\perp\xi).\end{aligned}$$

In particular, its shape operator A and its structure tensor field ϕ are simultaneously diagonalizable, i.e. $A\phi = \phi A$. Therefore, we find by (2.1) that the structure torsion of each trajectory for a Sasakian magnetic field on $G(r)$ is constant along the trajectory. When a trajectory has null structure torsion $\rho_\gamma = 0$, which is the case that the velocity vector $\dot\gamma$ is orthogonal to ξ, it is called a *Legendre* trajectory. We take the bundle $T^0G(r) = \{v \in TG(r) \mid v \perp \xi\}$ of the *contact distribution*, and consider its sphere subbundle $U^0G(r) = \{v \in T^0G(r) \mid \|v\| = 1\}$. Then, Legendre trajectories on $G(r)$ induce a flow $\mathbf{F}_\kappa\varphi_t^0 : U^0G(r) \to U^0G(r)$, which is the restriction of $\mathbf{F}_\kappa\varphi_t$ onto $U^0G(r)$. We shall call this the *Legendre magnetic flow* for a Sasakian magnetic field $\mathbf{F}_\kappa$.

We say two smooth flows φ_t, ψ_t on a differentiable manifold X are (smoothly) *conjugate* with each other if there exist a diffeomorphism $f : X \to X$ and a constant α satidfying $\psi_t = f^{-1}\circ\varphi_{\alpha t}\circ f$ for all t. Our main result is the following

Theorem. *Let $G(r)$ be a geodesic sphere of radius r in a complex projective space $\mathbf{C}P^n(c)$. The Legendre magnetic flow $\mathbf{F}_\kappa\varphi_t^0$ for an arbitrary Sasakian magnetic field $\mathbf{F}_\kappa$ is smoothly conjugate to the Legendre geodesic flow φ_t^0. More precisely, there is a diffeomorphism f_κ on $U^0G(r)$ satisfying*

$$\mathbf{F}_\kappa\varphi_t^0 = f_\kappa^{-1}\circ\varphi^0_{\sqrt{\kappa^2\sin^2(\sqrt{c}r/2)+c}\,t/\sqrt{c}}\circ f_\kappa.$$

Moreover, our result shows the following

Corollary. *Let $G(r)$ be a geodesic sphere of radius r in a complex projective space $\mathbf{C}P^n(c)$. Then arbitrary Legendre magnetic flows for Sasakian magnetic fields on $G(r)$ are conjugate to each other.*

3. Tangent bundles of geodesic spheres. In order to study Legendre trajectories on geodesic spheres in a complex projective space, we here recall the expression of the tangent bundle of a geodesic sphere through a Hopf fibration (cf. [9]). It is enough to study the case $c = 4$. Let $\varpi : S^{2n+1} \to \mathbf{C}P^n(4)$ denote a Hopf fibration of a unit sphere. The tangent space at $z = (z_0, \dots, z_n) \in S^{2n+1} \subset \mathbf{C}^{n+1}$, which is given as

$$T_zS^{2n+1} = \{(z,v)\in\{z\}\times\mathbf{C}^{n+1} \mid \mathrm{Re}(z_0\bar v_0 + \cdots + z_n\bar v_n) = 0\},$$

is decomposed into horizontal and vertical spaces $\mathcal{H}_z$, $\mathcal{V}_z$ given by

$$\begin{aligned}\mathcal{H}_z &= \{(z,v)\in\{z\}\times\mathbf{C}^{n+1} \mid z_0\bar v_0 + \cdots + z_n\bar v_n = 0\},\\ \mathcal{V}_z &= \{(z,\sqrt{-1}az)\in\{z\}\times\mathbf{C}^{n+1} \mid a\in\mathbf{R}\}.\end{aligned}$$

We take a geodesic sphere $M = G(r)$ in $\mathbf{C}P^n(4)$ whose inverse image $\widehat M = \varpi^{-1}(G(r))$ is given by

$$\widehat M = \{z = (z_0,\dots,z_n)\in\mathbf{C}^{n+1} \mid |z_0| = \cos r,\ |z_1|^2 + \cdots + |z_n|^2 = \sin^2 r\}.$$

We hence have

$$T_z\widehat M = \{(z,v)\in\{z\}\times\mathbf{C}^{n+1} \mid \mathrm{Re}(z_0\bar v_0) = 0,\ \mathrm{Re}(z_1\bar v_1 + \cdots + z_n\bar v_n) = 0\}.$$

For the sake of simplicity, we express a point $z \in \widehat M$

as $z=(z_0,z_*)\in \mathbf{C}\times\mathbf{C}^n$. Considering signatures of principal curvatures of M, we see that the horizontal lift $\widehat{\mathcal{N}}_z\in(T_z\widehat{M})^\perp\cap\mathcal{H}_z$ of the unit normal $\mathcal{N}_{\varpi(z)}$ at z is given as $\widehat{\mathcal{N}}_z=(z,(\tan rz_0,-\cot rz_*))\in\{z\}\times\mathbf{C}^{n+1}$.

We now give an expression of the horizontal lift of the bundle $T^0M=\{u\in TM \mid u\perp\xi\}$. We denote by $\hat{\xi}_z\in T_z\widehat{M}\cap\mathcal{H}_z$ the horizontal lift of the characteristic vector $\xi_{\varpi(z)}$ at z. It is given by $-\bar{J}\widehat{\mathcal{N}}_z$ with the complex structure $\bar{J}$ on $\mathbf{C}^{n+1}$. If $(z,(v_0,v_*))\in T_z\widehat{M}\cap\mathcal{H}_z$ is orthogonal to $\hat{\xi}_z$, as it is also orthogonal to $\widehat{\mathcal{N}}_z$, we have

$$\begin{aligned}\tan rz_0\bar{v}_0-\cot r(z_1\bar{v}_1+\cdots+z_n\bar{v}_n)&=0,\\ z_0\bar{v}_0+\cdots+z_n\bar{v}_n&=0,\end{aligned}$$

hence have

$$z_0\bar{v}_0=0, \tag{3.1}$$

$$z_1\bar{v}_1+\cdots+z_n\bar{v}_n=0. \tag{3.2}$$

Since $|z_0|=\cos r\ (\neq 0)$, we find $v_0=0$. Thus, by puting $T^0\widehat{M}=\{u\in T\widehat{M}\mid u\perp\hat{\xi}\}$ we have

$$T^0_z\widehat{M}\cap\mathcal{H}_z=\{(z,(0,v_*))\in\{z\}\times\mathbf{C}^{n+1} \mid z_1\bar{v}_1+\cdots+z_n\bar{v}_n=0\}.$$

4. Legendre trajectories on geodesic spheres. In [3], we gave explicit expressions of horizontal lifts of trajectories through a Hopf fibration. We quickly recall the expression for Legendre trajectories. Let γ be a trajectory for $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ in a complex projective space $\mathbf{C}P^n(4)$. Through an isometric embedding $\iota: G(r)\to\mathbf{C}P^n(4)$, we identify this curve with the curve $\iota\circ\gamma$ and regard it as a curve in $\mathbf{C}P^n(4)$. We take its horizontal lift $\hat{\gamma}$ through a Hopf fibration $\varpi: S^{2n+1}\to\mathbf{C}P^n(4)$ of a unit sphere, and regard it as a curve in $\mathbf{C}^{n+1}$. In order to study γ, we are enough to investigate $\hat{\gamma}$.

We denote by $\widehat{\mathcal{N}}$ a unit normal of $\widehat{M}=\varpi^{-1}(G(r))$ in S^{2n+1}, which is the horizontal lift of $\mathcal{N}$, and denote by $\overline{\mathcal{N}}$ the outward unit normal of S^{2n+1} in $\mathbf{C}^{n+1}$. Let $\overline{\nabla}$ be the Riemannian connection on $\mathbf{C}^{n+1}$. For arbitrary vector fields X,Y tangent to $M=G(r)$, denoting their horizontal lifts by $\widehat{X},\widehat{Y}$, we have

$$\begin{aligned}\overline{\nabla}_{\widehat{X}}\widehat{Y}&=\widehat{\widetilde{\nabla}_XY}+\langle X,JY\rangle\bar{J}\overline{\mathcal{N}}-\langle X,Y\rangle\overline{\mathcal{N}}\\&=\widehat{\nabla_XY}+\langle AX,Y\rangle\widehat{\mathcal{N}}\\&\quad+\langle X,JY\rangle\bar{J}\overline{\mathcal{N}}-\langle X,Y\rangle\overline{\mathcal{N}}.\end{aligned}$$

By using this relationship, we obtain

$$\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}}=\kappa(\bar{J}\dot{\hat{\gamma}}-\rho_\gamma\widehat{\mathcal{N}})+\langle A\dot{\gamma},\dot{\gamma}\rangle\widehat{\mathcal{N}}-\overline{\mathcal{N}}.$$

Since we have $\langle A\dot{\gamma},\dot{\gamma}\rangle=\lambda(1-\rho_\gamma^2)+\delta\rho_\gamma^2$ with $\lambda=\cot r$ and $\delta=2\cot 2r$, which is constant along γ, and have

$$\overline{\nabla}_{\dot{\hat{\gamma}}}\widehat{\mathcal{N}}=-\widehat{A\dot{\gamma}}=-\lambda(\dot{\hat{\gamma}}+\rho_\gamma\bar{J}\widehat{\mathcal{N}})+\delta\rho_\gamma\bar{J}\widehat{\mathcal{N}},$$

we find

$$\overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}}=\kappa\bar{J}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}}-(\langle A\dot{\gamma},\dot{\gamma}\rangle-\kappa\rho_\gamma)\widehat{A\dot{\gamma}}-\dot{\hat{\gamma}}.$$

Thus, when $\rho_\gamma=0$, the differential equation turns to

$$\overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}}=\kappa\bar{J}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}}-\frac{1}{\sin^2 r}\dot{\hat{\gamma}}.$$

Solving this differential equation, we have

$$\begin{aligned}\hat{\gamma}(t)=e^{\sqrt{-1}\kappa t/2}\Big\{&A\cos\frac{1}{2}\sqrt{\kappa^2+(4/\sin^2 r)}\,t\\&+B\sin\frac{1}{2}\sqrt{\kappa^2+(4/\sin^2 r)}\,t\Big\}+C\end{aligned}$$

with $A,B,C\in\mathbf{C}^{n+1}$. Considering initial condition

$$\hat{\gamma}(0)=z\in\widehat{M}\subset\mathbf{C}^{n+1},$$

$$\dot{\hat{\gamma}}(0)=(z,v),\quad \widehat{\mathcal{N}}_z=(z,w)\in\{z\}\times\mathbf{C}^{n+1},$$

we find

$$\begin{aligned}A&=\sin^2 r\,z-\sin r\cos r\,w,\\ B&=\frac{\sin r}{\sqrt{\kappa^2\sin^2 r+4}}\big(-\sqrt{-1}\kappa\sin^2 r\,z\\&\qquad+\sqrt{-1}\kappa\sin r\cos r\,w+2v\big),\\ C&=\cos^2 r\,z+\sin r\cos r\,w.\end{aligned}$$

We now study Legendre magnetic flows on $U^0G(r)$. We express the horizontal lift $\hat{\gamma}$ of a trajectory γ on $G(r)$ in $\mathbf{C}P^n(4)$ as $\hat{\gamma}(t)=(Z_0(t),Z_*(t))\in\mathbf{C}\times\mathbf{C}^n$ by regarding $\hat{\gamma}$ as a curve in $\mathbf{C}^{n+1}$. We put $\hat{\gamma}'(t)=(V_0(t),V_*(t))\in\mathbf{C}\times\mathbf{C}^n$. Similarly, we set $z=(z_0,z_*)$, $v=(v_0,v_*)$. We then have $w=(\tan r\,z_0,-\cot r\,z_*)$. By setting two functions $\mathfrak{c}_\kappa$, $\mathfrak{s}_\kappa$ by

$$\mathfrak{c}_\kappa(t)=\cos\frac{1}{2}\sqrt{\kappa^2+\frac{4}{\sin^2 r}}\,t,$$

$$\mathfrak{s}_\kappa(t)=\sin\frac{1}{2}\sqrt{\kappa^2+\frac{4}{\sin^2 r}}\,t,$$

we have

$$Z_0(t) \equiv z_0, \qquad V_0(t) \equiv 0,$$

$$Z_*(t) = e^{\sqrt{-1}\kappa t/2}\Biggl\{\biggl(\mathfrak{c}_\kappa(t) - \frac{\sqrt{-1}\kappa\sin r}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t)\biggr)z_* + \frac{2\sin r}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t)v_*\Biggr\},$$

$$V_*(t) = e^{\sqrt{-1}\kappa t/2}\Biggl\{-\frac{2}{\sin r\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t)z_* + \biggl(\mathfrak{c}_\kappa(t) + \frac{\sqrt{-1}\kappa\sin r}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t)\biggr)v_*\Biggr\}.$$

Here, we note that $\|Z_0(t)\| = \cos r$, $\|Z_*(t)\| = \sin r$ and $\|V_*(t)\| = 1$ by (3.2). Thus we can express $Z_*(t)$ and $V_*(t)$ as

$$\begin{pmatrix} {}^tZ_*(t)/\sin r \\ {}^tV_*(t) \end{pmatrix} = e^{\sqrt{-1}\kappa t/2}A_\kappa(t)\begin{pmatrix} {}^tz_*/\sin r \\ {}^tv_* \end{pmatrix} \tag{4.1}$$

with a matrix

$$A_\kappa(t) = \begin{pmatrix} a_{11}^{(\kappa)}(t)I_n & a_{12}^{(\kappa)}(t)I_n \\ a_{21}^{(\kappa)}(t)I_n & a_{22}^{(\kappa)}(t)I_n \end{pmatrix},$$

where

$$a_{11}^{(\kappa)}(t) = \biggl\{\mathfrak{c}_\kappa(t) - \frac{\sqrt{-1}\kappa\sin r}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t)\biggr\},$$

$$a_{12}^{(\kappa)}(t) = \frac{2}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t),$$

$$a_{21}^{(\kappa)}(t) = -\frac{2}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t),$$

$$a_{22}^{(\kappa)}(t) = \biggl\{\mathfrak{c}_\kappa(t) + \frac{\sqrt{-1}\kappa\sin r}{\sqrt{\kappa^2\sin^2 r+4}}\,\mathfrak{s}_\kappa(t)\biggr\}.$$

Here ${}^tw$ denotes the transposed vector of $w \in \mathbf{C}^n$ and I_n denotes the $n \times n$ identity matrix. If we set a unitary matrix P_κ as

$$P_\kappa(t) = \begin{pmatrix} p_{11}^{(\kappa)}I_n & p_{12}^{(\kappa)}I_n \\ p_{21}^{(\kappa)}I_n & p_{22}^{(\kappa)}I_n \end{pmatrix}$$

with

$$p_{11}^{(\kappa)} = \frac{-\sqrt{-1}(\sqrt{\kappa^2\sin^2 r+4} - \kappa\sin r)^{1/2}}{\sqrt{2}(\kappa^2\sin^2 r+4)^{1/4}},$$

$$p_{12}^{(\kappa)} = \frac{\sqrt{-1}(\sqrt{\kappa^2\sin^2 r+4} + \kappa\sin r)^{1/2}}{\sqrt{2}(\kappa^2\sin^2 r+4)^{1/4}},$$

$$p_{21}^{(\kappa)} = \frac{(\sqrt{\kappa^2\sin^2 r+4} + \kappa\sin r)^{1/2}}{\sqrt{2}(\kappa^2\sin^2 r+4)^{1/4}},$$

$$p_{22}^{(\kappa)} = \frac{(\sqrt{\kappa^2\sin^2 r+4} - \kappa\sin r)^{1/2}}{\sqrt{2}(\kappa^2\sin^2 r+4)^{1/4}},$$

we obtain

$$\begin{pmatrix} {}^tZ_*(t)/\sin r \\ {}^tV_*(t) \end{pmatrix} = e^{\sqrt{-1}\kappa t/2} \times P_\kappa\begin{pmatrix} d_{11}^{(\kappa)}(t)I_n & O_n \\ O_n & d_{22}^{(\kappa)}(t)I_n \end{pmatrix}P_\kappa^*\begin{pmatrix} {}^tz_*/\sin r \\ {}^tv_* \end{pmatrix}, \tag{4.2}$$

where P_κ^* denotes the adjoint matrix of P_κ and

$$d_{11}^{(\kappa)}(t) = \{\mathfrak{c}_\kappa(t) + \sqrt{-1}\mathfrak{s}_\kappa(t)\},$$

$$d_{22}^{(\kappa)}(t) = \{\mathfrak{c}_\kappa(t) - \sqrt{-1}\mathfrak{s}_\kappa(t)\}.$$

By identifying $T\mathbf{C}^{n+1}$ to $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1} = \mathbf{C} \times \mathbf{C}^n \times \mathbf{C} \times \mathbf{C}^n$, we define a diffeomorphism $\hat{f}_\kappa : T\mathbf{C}^{n+1} \to T\mathbf{C}^{n+1}$ by $\widehat{Q}\widehat{P}_0\widehat{P}_\kappa^*\widehat{Q}^{-1}$, where

$$\widehat{P}_\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_{11}^{(\kappa)}I_n & 0 & p_{12}^{(\kappa)}I_n \\ 0 & 0 & 1 & 0 \\ 0 & p_{21}^{(\kappa)}I_n & 0 & p_{22}^{(\kappa)}I_n \end{pmatrix},$$

$$\widehat{P}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{-2}\,I_n & 0 & \sqrt{-1/2}\,I_n \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{2}\,I_n & 0 & 1/\sqrt{2}\,I_n \end{pmatrix},$$

$$\widehat{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin rI_n & 0 & O_n \\ 0 & 0 & 1 & 0 \\ 0 & O_n & 0 & I_n \end{pmatrix}.$$

This diffeomorphism preserves the bundle $\bigcup_{z\in\widehat{M}}(U_z^0\widehat{M} \cap \mathcal{H}_z)$ over $\widehat{M} = \varpi^{-1}(G(r))$ by regarding it as a subbundle of $\widehat{M} \times \mathbf{C}^{n+1}$ $(\subset T\mathbf{C}^{n+1})$. Here, $U_z^0\widehat{M}$ denotes the set of all unit tangent vectors of $\widehat{M}$ at z which are orthogonal to $\hat{\xi}_z$. Clearly it satisfies $\hat{f}_\kappa(\sqrt{-1}\eta) = \sqrt{1}\hat{f}_\kappa(\eta)$ for every $\eta \in T\mathbf{C}^{n+1}$. We therefore obtain a diffeomorphism $f_\kappa : U^0G(r) \to U^0G(r)$ satisfying $d\varpi \circ \hat{f}_\kappa = f_\kappa \circ d\varpi$. Since we have

$$\mathbf{F}^0_\kappa\varphi_t\left(d\varpi\left(\begin{pmatrix} {}^t z \\ {}^t v \end{pmatrix}\right)\right)$$
$$= d\varpi\left(e^{\sqrt{-1}\kappa t/2}\widehat{Q}\widehat{A}_\kappa(t)\widehat{Q}^{-1}\begin{pmatrix} {}^t z \\ {}^t v \end{pmatrix}\right)$$
$$= d\varpi\left(\widehat{Q}\widehat{A}_\kappa(t)\widehat{Q}^{-1}\begin{pmatrix} {}^t z \\ {}^t v \end{pmatrix}\right)$$

by (4.1) with the matrix

$$\widehat{A}_\kappa(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^{(\kappa)}_{11}(t)I_n & 0 & a^{(\kappa)}_{12}(t)I_n \\ 0 & 0 & 1 & 0 \\ 0 & a^{(\kappa)}_{21}(t)I_n & 0 & a^{(\kappa)}_{22}(t)I_n \end{pmatrix},$$

we find that the expression (4.2) shows

$$\mathbf{F}_\kappa\varphi^0_t = f_\kappa^{-1}\circ\varphi^0_{\sqrt{\kappa^2\sin^2 r+4}\,t/2}\circ f_\kappa.$$

In order to study Legendre magnetic flows for $G(r)$ in $\mathbf{C}P^n(c)$, we change the metric homothetically. By using the Riemannian metric $\langle\,,\rangle$ on $\mathbf{C}P^n(c)$ we define a new metric $\langle\,,\rangle'$ by $\langle\,,\rangle' = (\sqrt{c}/2)\langle\,,\rangle$. With respect to this new metric, the geodesic sphere $G(r)$ can be seen as a geodesic sphere $G(r')$ of radius $r' = \sqrt{c}\,r/2$ in $\mathbf{C}P^n(4)$. When γ is a trajectory for $\mathbf{F}_\kappa$ on $G(r)$, then the curve $\sigma(s) = \gamma(2s/\sqrt{c})$ is a trajectory for $\mathbf{F}_{\kappa'}$ with $\kappa' = 2\kappa/\sqrt{c}$ on $G(r')$. We denote by $(\mathbf{F}^0_{\kappa'}\varphi_t)'$ and $(\varphi^0_t)'$ the Legendre magnetic flow for $\mathbf{F}_{\kappa'}$ and the Legendre geodesic flow for $G(r')$, respectively. By using a diffeomorphism $f'_{\kappa'}$ of $U^0G(r')$, we define a diffeomorphism f_κ of $U^0G(r)$ by $f_\kappa(v) = (\sqrt{c}/2)\,f_{\kappa'}((2/\sqrt{c})v)$. We then have

$$\mathbf{F}_\kappa\varphi^0_t(v) = \frac{\sqrt{c}}{2}\,(\mathbf{F}_{\kappa'}\varphi^0_{\sqrt{c}t/2})'\left(\frac{2}{\sqrt{c}}v\right)$$
$$= \frac{\sqrt{c}}{2}\,(f'_{\kappa'})^{-1}\circ(\varphi^0_{\sqrt{(\kappa')^2\sin^2 r'+4}\,\sqrt{c}t/4})'\circ f'_{\kappa'}\left(\frac{2}{\sqrt{c}}v\right)$$
$$= f_\kappa^{-1}\circ\varphi^0_{\sqrt{\kappa^2\sin^2(\sqrt{c}r/2)+c}\,t/\sqrt{c}}\circ f_\kappa(v).$$

This shows our theorem.

At last, we compare our result to that on Kähler magnetic flows on a complex projective space. A constant multiple $\mathbf{B}_\kappa = \kappa\mathbf{B}_J$ of the Kähler form $\mathbf{B}_J$ on a Kähler manifold M is said to be a Kähler magnetic field. Hence its trajectory γ is a smooth curve parameterized by its arclength satisfying the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Trajectories for $\mathbf{B}_\kappa$ induce a Kähler magnetic flow $\mathbf{B}_\kappa\varphi_t$ on the unit tangent bundle UM of M. In [1], it was shown that every Kähler magnetic flow $\mathbf{B}_\kappa\varphi_t$ on a complex projective space is smoothly conjugate to the geodesic flow φ_t. That is, there is a diffeomorphism $g_\kappa : U\mathbf{C}P^n(c)\to U\mathbf{C}P^n(c)$ satisfying $\mathbf{B}_\kappa\varphi_t = g_\kappa^{-1}\circ\varphi_{\sqrt{\kappa^2+c}\,t/\sqrt{c}}\circ g_\kappa$. We should note that a geodesic sphere $G(r)$ in $\mathbf{C}P^n(c)$ has constant ϕ-sectional curvature $c(4+\cot^2(\sqrt{c}\,r/2))/4$ (see [8]).

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References

[1] T. Adachi, Kähler magnetic fields on a complex projective space, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), no. 1, 12–13.
[2] T. Adachi, Kähler magnetic flows for a manifold of constant holomorphic sectional curvature, Tokyo J. Math. **18** (1995), no. 2, 473–483.
[3] T. Adachi, Trajectories on geodesic spheres in a non-flat complex space form, J. Geom. **90** (2008), no. 1–2, 1–29.
[4] T. Adachi and S. Maeda, Length spectrum of complete simply connected Sasakian space forms, Differential Geom. Appl. **70** (2020), paper No. 101625.
[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, 2nd edition Progr. Math., 203 Birkhäuser Boston, Ltd., Boston, MA, 2010.
[6] J. L. Cabrerizo, M. Fernández and J. S. Gómez, The contact magnetic flow in 3D Sasakian manifolds, J. Phys. A **42** (2009), no. 19, paper No. 195201.
[7] S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor, Magnetic curves in Sasakian manifolds, J. Nonlinear Math. Phys. **22** (2015), no. 3, 428–447.
[8] S. Maeda and M. Kimura, Sectional curvatures of some homogeneous real hypersurfaces in a complex projective space, in *Topics in contemporary differential geometry, complex analysis and mathematical physics*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007, pp. 196–204.
[9] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, in *Tight and taut submanifolds*, Math. Sci. Res. Inst. Publ., 32, Cambridge University Press, Cambridge, 1997, pp. 233–305.

[10] Q. Shi and T. Adachi, A note on Legendre trajectories on Sasakian space forms, in *New horizons in differential geometry and its related fields*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2022, pp. 41–50.
[11] T. Sunada, Magnetic flows on a Riemann surface, Proc. KAIST Math., Workshop **8** (1993), 93–108.