132. Probability-theoretic Investigations on Inheritance. IV₅. Mother-Child Combinations.

(Further Continuation.)

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(Comm. by T. FURUHATA, M.J.A., Nov. 12, 1951.)

5. Mother-child-child combination.

The object of the discussions done in §3 of IV concerned, as stated there explicitly, two children belonging to the same family and their mother, namely two children having both parents in common and their mother. Besides mother-children combinations of this sort, there is an another sort of combinations consisting of a mother and of her two children not having a father in common. Such a combination will occur, for instance, when a mother who was divorced by or separated by death from her former husband has married again bringing a child and then produces a new child with her present husband. In the present section we shall consider *mother-child-child combinations* of this sort.

We treat rather generally, as in §2 of IV, the case of mixed combinations. Let a mother belong, as usual, to a population with distribution $\{p_i\}$, and let fathers of the first and second children to populations with distributions $\{p'_i\}$ and $\{p''_i\}$, respectively. In particular, if

$$(5.1) \qquad\qquad p_i = p''_i \qquad\qquad (i = 1, \ldots, m)$$

or if

$$(5.2) \qquad\qquad p'_i = p_i \qquad\qquad (i = 1, \ldots, m),$$

the results reduce to those in case where both fathers or the father of the first child and the mother belong to the same population, respectively, and if further

$$(5.3) \qquad\qquad p'_i = p''_i = p_i \qquad\qquad (i = 1, \ldots, m),$$

then the reduced case appears where both fathers belong all to the same population as the mother.

Now, let the probability of mother-child combination consisting of a mother A_{ij} and her first child A_{hk} be denoted by

(5.4) $$\pi'(ij\,;\,hk)$$

in coformity with the notation used in §2 of IV. Let that of mother-child combination consisting of a mother A_{ij} and the second child A_{fg} be similarly denoted by

(5.5) $$\pi''(ij\,;\,fg).$$

It is evident, that $\pi''(ij\,;\,fg)$ may be obtained from $\pi'(ij\,;\,fg)$ by substituting p'''s instead of the corresponding p''s.

Let further *the probability of mother-child-child combination consisting of a mother* A_{ij}, *her first child* A_{hk} *and her second child* A_{fg} *be denoted by*

(5.6) $$\pi^*(ij\,;\,hk,\,fg),$$

an *agreement* corresponding to that immediately subsequent to (3.5) of IV being supposed to be here also made.

For a fixed type A_{ij} of mother, the probability of the event that her first and second children are of the types A_{hk} and A_{fg} respectively is equal to

$$\frac{\pi'(ij\,;\,hk)}{\bar{A}_{ij}}\cdot\frac{\pi''(ij\,;\,fg)}{\bar{A}_{ij}},$$

$\bar{A}_{ij}$ being the distribution-probability of the genotype A_{ij} in the population containing the mother, namely

$$\bar{A}_{ij}=\begin{cases}p_i^2 & (i=j),\\ 2p_ip_j & (i\neq j).\end{cases}$$

We get, therefore, the *fundamental relation*

(5.7) $$\pi^*(ij\,;\,hk,\,fg)=\pi'(ij\,;\,hk)\,\pi''(ij\,;\,fg)/\bar{A}_{ij}\,,$$

by means of which, taking the table in §2 of IV into account, we can construct the table on mother-child-child combination. In the table, the same agreement is made as in that in §3 of IV.

Mother	First child	Prob. of comb. betw. mother and first child	Second child			
			A_{ii}	A_{ih}	A_{ik}	A_{fg}
A_{ii}	A_{ii}	$p_i^2p_i'$	$p_i^2p_i'p_i''$	$p_i^2p_i'p_h''$	$p_i^2p_i'p_k''$	0
	A_{ih}	$p_i^2p_h'$	$p_i^2p_h'p_i''$	$p_i^2p_h'p_h''$	$p_i^2p_h'p_k''$	0
	A_{ik}	$p_i^2p_k'$	$p_i^2p_k'p_i''$	$p_i^2p_k'p_h''$	$p_i^2p_k'p_k''$	0
	A_{fg}	0	0	0	0	0
		p_i^2	$p_i^2p_i''$	$p_i^2p_h''$	$p_i^2p_k''$	0

Mothers	First child	Prob. of comb. betw. mother and first child	A_{ii}	A_{jj}	A_{ij}	A_{ih}	A_{jh}	A_{ik}	A_{jk}	A_{fg}
					Second child					
A_{ij}	A_{ii}	$p_ip_jp_i'$	$\frac{1}{2}p_ip_jp_i'p_i''$	$\frac{1}{2}p_ip_jp_i'p_j''$	$\frac{1}{2}p_ip_jp_i'(p_i''+p_j'')$	$\frac{1}{2}p_ip_jp_i'p_h''$	$\frac{1}{2}p_ip_jp_i'p_h''$	$\frac{1}{2}p_ip_jp_i'p_k''$	$\frac{1}{2}p_ip_jp_i'p_k''$	0
	A_{jj}	$p_ip_jp_j'$	$\frac{1}{2}p_ip_jp_j'p_i''$	$\frac{1}{2}p_ip_jp_j'p_j''$	$\frac{1}{2}p_ip_jp_j'(p_i''+p_j'')$	$\frac{1}{2}p_ip_jp_j'p_h''$	$\frac{1}{2}p_ip_jp_j'p_h''$	$\frac{1}{2}p_ip_jp_j'p_k''$	$\frac{1}{2}p_ip_jp_j'p_k''$	0
	A_{ij}	$p_ip_j(p_i'+p_j')$	$\frac{1}{2}p_ip_j(p_i'+p_j')p_i''$	$\frac{1}{2}p_ip_j(p_i'+p_j')p_j''$	$\frac{1}{2}p_ip_j(p_i'+p_j')(p_i''+p_j'')$	$\frac{1}{2}p_ip_j(p_i'+p_j')p_h''$	$\frac{1}{2}p_ip_j(p_i'+p_j')p_h''$	$\frac{1}{2}p_ip_j(p_i'+p_j')p_k''$	$\frac{1}{2}p_ip_j(p_i'+p_j')p_k''$	0
	A_{ih} or A_{jh}	$p_ip_jp_h'$	$\frac{1}{2}p_ip_jp_h'p_i''$	$\frac{1}{2}p_ip_jp_h'p_j''$	$\frac{1}{2}p_ip_jp_h'(p_i''+p_j'')$	$\frac{1}{2}p_ip_jp_h'p_h''$	$\frac{1}{2}p_ip_jp_h'p_h''$	$\frac{1}{2}p_ip_jp_h'p_k''$	$\frac{1}{2}p_ip_jp_h'p_k''$	0
	A_{ik} or A_{jk}	$p_ip_jp_k'$	$\frac{1}{2}p_ip_jp_k'p_i''$	$\frac{1}{2}p_ip_jp_k'p_j''$	$\frac{1}{2}p_ip_jp_k'(p_i''+p_j'')$	$\frac{1}{2}p_ip_jp_k'p_h''$	$\frac{1}{2}p_ip_jp_k'p_h''$	$\frac{1}{2}p_ip_jp_k'p_k''$	$\frac{1}{2}p_ip_jp_k'p_k''$	0
	A_{hk}	0	0	0	0	0	0	0	0	0
		$2p_ip_j$	$p_ip_jp_i''$	$p_ip_jp_j''$	$p_ip_j(p_i''+p_j'')$	$p_ip_jp_h''$	$p_ip_jp_h''$	$p_ip_jp_k''$	$p_ip_jp_k''$	0

It will be evident that the identities

(5.8)
$$\sum_{f \leq g} \pi^*(ij \; ; \; hk, fg) = \pi'(ij \; ; \; hk) \, ,$$

$$\sum_{h \leq k} \pi^*(ij \; ; \; hk, fg) = \pi''(ij \; ; \; fg)$$

hold good, since, as noticed in (2.13) of IV,

$$\sum_{f \leq g} \pi''(ij \; ; \; fg) = \sum_{h \leq k} \pi'(ij \; ; \; hk) = \bar{A}_{ij} \, .$$

The passage to the results on phenotypes can be done by the usual procedure. The formulae analogous to those such as (3.27) of IV will be derived.

As already noticed, the specialization (5.3) leads to the results in the case where both fathers belong to the same population as mother. We shall now compare the thus obtained results with those on mother-children combination discussed in §3 of IV.

—To be continued—