89. On the Adiabatic Theorem for the Hamiltonian System of Differential Equations in the Classical Mechanics. II

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3. Let (X, m) be a measure space where m is a finite, separable, and complete measure[1] defined on a Borel field in X. A one-parameter group $\{\mathfrak{T}_t \mid -\infty < t < +\infty\}$ of one-to-one mappings $\mathfrak{T}_t$ of X onto X is called a flow on (X, m). A measurable function $f(P)$ on (X, m) is called an invariant function of a flow $\{\mathfrak{T}_t\}$ on (X, m) if

$$f(\mathfrak{T}_t(P)) = f(P)$$

almost everywhere on (X, m) for every fixed t and it is called a strictly invariant function of a flow $\{\mathfrak{T}_t\}$ on (X, m) if it is defined everywhere on X and

$$f(\mathfrak{T}_t(P)) = f(P)$$

for all (P, t) such that $P \in X, -\infty < t < +\infty$. A measure-preserving and measurable flow[2] $\{\mathfrak{T}_t\}$ on (X, m) is ergodic (in the sense of J. v. Neumann) if and only if all its invariant functions are equivalent[3] to constants on (X, m). If a flow $\{\mathfrak{T}_t\}$ on (X, m) is measure-preserving and measurable, then we can associate with it a one-parameter group $\{\mathfrak{U}_t \mid -\infty < t < +\infty\}$ of unitary transformations $\mathfrak{U}_t$ on $L^2(X, m)$ by

$$(\mathfrak{U}_t f)(P) = f(\mathfrak{T}_t(P)) \quad f \in L^2(X, m), \quad P \in X$$

and $\mathfrak{U}_t$ is continuous as a function of t in the strong topology of $\mathfrak{U}_t$.[4]

If X is a Lebesgue measurable subset of a Euclidean space R^r and m is the usual Lebesgue measure in R^r defined for all Lebesgue measurable subsets of X, a flow on (X, m) is simply called a flow on X in the following and we write simply $L^2(X)$ for $L^2(X, m)$.

4. We consider the Hamiltonian system with a parameter s

$$(9) \qquad dp/dt = -\partial H/\partial q(p, q, s) \quad dq/dt = \partial H/\partial p(p, q, s).$$

By Assumption 1, the solution of (9)

$$(10) \qquad p = \mathfrak{p}(t, p^0, q^0, s) \quad q = \mathfrak{q}(t, p^0, q^0, s)$$

in the open set $I(s)$ for a fixed s $(a \leqq s \leqq b)$ with the initial conditions $(p, q) = (p^0, q^0)((p^0, q^0) \in I(s))$ at $t = 0$, can be uniquely prolonged for the

1) For the definition of complete or separable measure, cf. P. Halmos [1].

2) For the definition of a measure-preserving, a measurable or an ergodic flow on (X, m), cf. E. Hopf [2, pp. 8–9 and p. 28].

3) Two measurable functions on (X, m) are called equivalent on (X, m) if they coincide almost everywhere on (X, m).

4) For definitions and results concerning flows on a measure space used in this paper, cf. E. Hopf [2].

whole time interval $-\infty<t<+\infty$ and $(\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s)) \in S(J, s)$ for $-\infty<t<+\infty$ if $(p^0, q^0)\in S(J, s)$, since $H(p, q, s)$ is an integral of (9), $S(J, s)=\{(p, q) \mid H(p, q, s)=\mathfrak{E}(J, s), (p, q)\in G_s\}$ and $S(J, s)$ is compact for $J_2^*>J>J_1^*, b\geqq s\geqq a$.[5] Also $\mathfrak{p}_i(t, p^0, q^0, s), \mathfrak{q}_i(t, p^0, q^0, s) \in C^1[(-\infty, +\infty)\times D]$ $i=1,\cdots, n$.[5] We denote by $T_t^{(s)}$ the one-to-one mapping of $I(s)$ onto $I(s)$

(11) $$(p^0, q^0)\rightarrow(\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s)).$$

Then $\{T_t^{(s)} \mid -\infty<t<+\infty\}$ constitutes a flow F_s on $I(s)$ for $a\leqq s\leqq b$, since the right sides of (9) do not contain the time t explicitly. The flow F_s on $I(s)$ is measure-preserving and measurable since (9) is a Hamiltonian system[6] (Theorem of Liouville) and $\mathfrak{p}_i(t, p^0, q^0, s)$, $\mathfrak{q}_i(t, p^0, q^0, s)$ $(i=1,\cdots, n)$ has sufficient regularities.

Since $T_t^{(s)}$ transforms $S(J, s)$ onto $S(J, s)$ $T_t^{(s)}$ induces a one-to-one mapping $T_t^{(J,s)}$ of $S(J, s)$ onto $S(J, s)$. $\{T_t^{(J,s)} \mid -\infty<t<+\infty\}$ constitutes a flow $F_{J,s}$ on the measure space $(S(J, s), m_{J,s})$ for $J_2^*>J>J_1^*, b\geqq s\geqq a$.

LEMMA 2. *The flow $F_{J,s}$ on $(S(J, s), m_{J,s})$ is measure-preserving and measurable for $J_2^*>J>J_1^*, b\geqq s\geqq a$.*

We shall give a proof of this lemma in Part IV of this paper.

Also we consider the one-to-one mapping T_t of D onto D defined by

(12) $$(p^0, q^0, s)\rightarrow(\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s), s).$$

$\{T_t \mid -\infty<t<+\infty\}$ constitutes a flow F on D. From the fact that the flow F_s on $I(s)$ is measure-preserving and $\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s)$ are sufficiently regular, it follows easily that the flow F on D is measure-preserving and measurable. Then we have

THEOREM 2. *For any fixed s $(b\geqq s\geqq a)$ the two following conditions* i) *and* ii) *are equivalent:*

i) *Every invariant function $f(p, q)$ of the flow F_s on $I(s)$ is equivalent on $I(s)$ to a function of the form $\varphi(H(p, q, s))$ where $\varphi(E)$ is a measurable function of E for the interval $\mathfrak{E}(J_2^*, s)>E>\mathfrak{E}(J_1^*, s)$.*

ii) *The flow $F_{J,s}$ on $(S(J, s), m_{J,s})$ is ergodic for almost all J in the interval $J_2^*>J>J_1^*$.*

We shall give a proof of this theorem in Part IV. This theorem is not used for the proof of our main theorem (the adiabatic theorem). It is laid here only to clarify the meaning of the following Assumption 3.

5. Now we put a further

ASSUMPTION 3. *The condition* i) *in Theorem* 2 (*equivalent to the condition* ii) *in Theorem* 2) *is satisfied at almost all s in the interval $a\leqq s\leqq b$.*

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5) Cf. E. Kamke [3, pp. 135–136 and pp. 161–164].

6) Cf. E. Kamke [3, pp. 155–161].

We can easily prove that Assumption 3 is also equivalent to the following proposition: For almost all s in the interval $a\leqq s\leqq b$, every invariant function $f(p,q)$ of the flow F_s on $I(s)$ is equivalent on $I(s)$ to a function of the form $\psi(\tilde{\mathfrak{J}}(p,q,s))$ where $\psi(J)$ is a measurable function of J for the interval $J_2^*>J>J_1^*$.

LEMMA 3. *If an invariant function $f(p,q,s)$ of the flow F on D belongs to $L^2(D)$, then*

$$\int_D \overline{f(p,q,s)}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}(p,q,s)dpdqds=0.^{7)}$$

PROOF. We can assume that $f(p,q,s)$ is a strictly invariant function of the flow F on D since for every invariant function of the flow F on D, there is a strictly invariant function of the flow F on D equivalent to it on D.[8] Then for almost all s in the interval $b\geqq s\geqq a$, $f(p,q,s)$ as a function of (p,q) is an invariant function of the flow F_s on $I(s)$ and so by Assumption 3 is equivalent on $I(s)$ to a function of the form $\psi_s(\tilde{\mathfrak{J}}(p,q,s))$ where $\psi_s(J)$ is a measurable function of J on the interval $J_2^*>J>J_1^*$.

Hence

$$\begin{aligned}\int_D \bar{f}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dpdqds&=\int_a^b\Big(\int_{I(s)}\bar{f}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dpdq\Big)ds\\&=\int_a^b\Big\{\int_{J_1^*}^{J_2^*}\Big(\int_{S(J,s)}\overline{\psi_s(\tilde{\mathfrak{J}})}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ\Big\}ds\\&=\int_a^b\Big\{\int_{J_1^*}^{J_2^*}\overline{\psi_s(J)}\Big(\int_{S(J,s)}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ\Big\}ds=0\end{aligned}$$

by Fubini's Theorem, Lemma 1, and Theorem 1. Q. E. D.

Now we consider the one-parameter group $\{U_t\}$ of unitary transformations on $L^2(D)$ associated with the flow F on D. We define Af by

$$\left\|\frac{U_tf-f}{it}-Af\right\|_D\overset{9)}{\to}0\quad(t\to0)\tag{13}$$

for all $f\in L^2(D)$ for which such Af exists. Then by a theorem of Stone, A is a self-adjoint operator (in the sense of J. v. Neumann) on $L^2(D)$ and $U_t=e^{iAt}$ in the sense of the operator calculus.[10] We denote the domain and the range of A by $\mathfrak{D}(A)$ and by $\mathfrak{R}(A)$ respectively. For a function $f\in C_0^1(D^0)$, we define $f=0$ on $D-D^0$[11] for

7) Here the bar means the complex conjugate.

8) Cf. E. Hopf [2, pp. 27-28].

9) $\|\ \ \|_D$ means the norm in $L^2(D)$.

10) Cf. F. Riesz and B. Sz.-Nagy [4, pp. 383-385].

11) If we denote for each s the set $(p,q,s)\mid(p,q)\in I(s)\}$ by $\tilde{I}(s)$, then $D-D^0=\tilde{I}(a)\cup\tilde{I}(b)$ since D is relatively open in K.

convenience sake in the following. Then $C_0^1(D^0)\subset L^2(D)$ and also $C_0^1(D^0)\subset C^1(D)$. By calculating explicitly the Af in (13) for $f\in C_0^1(D^0)$, we have easily

LEMMA 4. *If $f\in C_0^1(D^0)$, then $f\in \mathfrak{D}(A)$ and*

$$Af=-i(f,H)=i(H,f).^{12)}$$

Now we prove a lemma which is useful for some applications of Stone's Theorem.

LEMMA 5. *Let L be a self-adjoint operator on an abstract complex Hilbert space $\mathfrak{H}$. Let $\{V_t \mid -\infty<t<+\infty\}$ be the strongly continuous one-parameter group of unitary transformations $V_t=e^{iLt}$ on $\mathfrak{H}$. We put $\mathfrak{N}=\{f \mid f\in\mathfrak{D}(L),\ Lf=0\}$. Then $\mathfrak{N}$ is a closed linear subspace of $\mathfrak{H}$. We denote by $\mathfrak{N}^\perp$ the orthogonal complement of $\mathfrak{N}$ in $\mathfrak{H}$. Let $\mathfrak{W}$ be a linear (not necessarily closed) subspace of $\mathfrak{H}$, invariant for the group V_t, that is, such that, $V_t(\mathfrak{W})=\mathfrak{W}$ for all t. Also let $\mathfrak{W}\subset\mathfrak{D}(L)$ and $\overline{\mathfrak{W}}=\mathfrak{H}$. Then we have $\overline{L(\mathfrak{W})}=\mathfrak{N}^\perp$.*[13]

PROOF. $\overline{\mathfrak{R}(L)}=\mathfrak{N}^\perp$ since L is self-adjoint in $\mathfrak{H}$. Hence $\overline{L(\mathfrak{W})}\subset\mathfrak{N}^\perp$. Also $\overline{L(\mathfrak{W})}$ is a closed linear subspace of $\mathfrak{H}$. Let us assume that $\overline{L(\mathfrak{W})}\neq\mathfrak{N}^\perp$. Then there exists an element $f\in\mathfrak{N}^\perp$ such that $f\neq 0$ and $(f,Lg)=0$ for all $g\in\mathfrak{W}$. Now we have for all $g\in\mathfrak{W}$ and for all t

$$\frac{d}{dt}(f,V_tg)=\lim_{\Delta t\to 0}\left(f,\frac{V_{\Delta t}(V_tg)-V_tg}{\Delta t}\right)=(f,iLV_tg)$$

since $V_tg\in\mathfrak{W}\subset\mathfrak{D}(L)$[14] by $g\in\mathfrak{W}$, $V_t(\mathfrak{W})=\mathfrak{W}$ and $\mathfrak{W}\subset\mathfrak{D}(L)$. From this, we have for all $g\in\mathfrak{W}$ and for all t

$$\frac{d}{dt}(f,V_tg)=0$$

since $(f,iLV_tg)=0$ by the assumed properties of f and $V_tg\in\mathfrak{W}$.

Therefore for each $g\in\mathfrak{W}$, (f,V_tg) and so $(V_tf,g)\ (=(f,V_{-t}g))$ are constants for $-\infty<t<+\infty$ so that $(V_tf-f,g)=(V_tf-V_0f,g)=0$ for all $g\in\mathfrak{W}$ and for all t. Hence $V_tf=f$ for all t since $\overline{\mathfrak{W}}=\mathfrak{H}$. From this, it follows easily that $f\in\mathfrak{D}(L)$ and $Lf=0$[14] so that $f\in\mathfrak{N}$. Hence $f=0$ since also $f\in\mathfrak{N}^\perp$ by the assumption. This contradicts the assumption that $f\neq 0$. Q. E. D.

We return to the discussion of the group $\{U_t\}$ of unitary transformations associated with the flow F. We denote by N the set of all invariant functions of the flow F on D belonging to $L^2(D)$. N coincides with the set $\{f \mid f\in L^2(D),\ f=U_tf\ \ -\infty<t<+\infty\}=\{f \mid f\in\mathfrak{D}(A),\ Af=0\}$.[15] Hence N is a closed linear subspace of $L^2(D)$. Also

12) If $f,g\in C^1(D)$, we denote by (f,g) the Poisson bracket $-\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right)$.

13) $\mathfrak{D}(L)$ and $\mathfrak{R}(L)$ are the domain of definition and range of L respectively. $L(\mathfrak{W})$ and $V_t(\mathfrak{W})$ are the images of $\mathfrak{W}$ by L and V_t respectively.

14) Cf. F. Riesz and B. Sz.-Nagy [4, pp. 383–385].

15) The last identy follows easily from the spectral representations of $\{U_t\}$ and A. Cf. F. Riesz and B. Sz.-Nagy [4, pp. 383–385].

we denote the orthogonal complement of N in $L^2(D)$ by $N^{\perp}$. Then we have

LEMMA 6. *$A[C_0^1(D^0)]$, the image of $C_0^1(D^0)$ by A, is contained and is dense in $N^{\perp}$.*

PROOF. $C_0^1(D^0)$ is dense in $L^2(D)$ as is well known. Also by Lemma 4 we have $C_0^1(D^0)\subset \mathfrak{D}(A)$. Further we have $U_t(C_0^1(D^0))=C_0^1(D^0)$ for all $t(-\infty<t<+\infty)$ since $\mathfrak{p}_i(t, p^0, q^0, s)$ and $\mathfrak{q}_i(t, p^0, q^0, s)(i=1, \cdots, n)$ all belong to $C^1[(-\infty, +\infty)\times D]$ and $T_t(D^0)=D^0$ for all t. Therefore we get the desired result by Lemma 5.

Now we prove the most important lemma of this Part II.

LEMMA 7. *For any $\varepsilon>0$, there is a function $f_0(p, q, s)\in C_0^1(D^0)$ such that*

$$\left\| (H,f_0)-\frac{\partial\tilde{\mathfrak{J}}}{\partial s} \right\|_D<\varepsilon.$$

PROOF. By Lemma 3, we have $\partial\tilde{\mathfrak{J}}/\partial s\in N^{\perp}$, if $\partial\tilde{\mathfrak{J}}/\partial s$ is considered as an element of $L^2(D)$. Hence by Lemma 6, for any $\varepsilon>0$, we have a function $f_0'\in C_0^1(D^0)$ such that

$$\left\| Af_0'-\frac{\partial\tilde{\mathfrak{J}}}{\partial s} \right\|_D<\varepsilon.$$

Therefore if we put $f_0=if_0'$, we get the desired result, since we have

$$Af_0=i(H,f_0)$$

for $f_0\in C_0^1(D^0)$, by Lemma 4. Q. E. D.

References

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