85. A Note on Hausdorff Spaces with the Star-finite Property. III

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We shall prove in this note, by a very simple argument, that an arbitrary non-empty (separable) metric space R is the image of a 0-dimensional (separable) metric space, under the open continuous mapping. At the first sight this is an odd fact, in view of Yu. Rozanskaya's theorem [3] which asserts that there does not exist an open continuous mapping of an m-dimensional Euclidean cube R_m onto an n-dimensional Euclidean cube R_n with $m<n$.

Theorem 1. *A topological T_1-space R is always the image of a completely regular space A with* $\operatorname{ind} A=0$ *under the open continuous mapping f such that $f^{-1}(x)$ is compact for every point x of R.*

Proof. Let $\{\mathfrak{U}_\lambda=\{U_\alpha; \alpha\in A_\lambda\}; \lambda\in\Lambda\}$ be a family of all finite open coverings of R. Let A be the aggregate of points $a=(\alpha_\lambda; \lambda\in\Lambda)$ of the product space $\Pi\{A_\lambda; \lambda\in\Lambda\}$, where A_λ are topological spaces with the discrete topology, such that $\frown\{U_{\alpha_\lambda}; \lambda\in\Lambda\}\neq\phi$. Let $f(a)=\frown\{U_{\pi_\lambda(a)}; \lambda\in\Lambda\}$, where $\pi_\lambda: A\to A_\lambda$, $\lambda\in\Lambda$, are the projections. Then f is a mapping of A onto R. Since for any $\lambda\in\Lambda$ and any $\alpha\in A_\lambda$ we have $f(\pi_\lambda^{-1}(\alpha))=U_\alpha$, f is an open continuous mapping. Let x be an arbitrary point of R and $B_\lambda=\{\alpha; x\in U_\alpha\in\mathfrak{U}_\lambda\}$, $\lambda\in\Lambda$. Then $f^{-1}(x)=\Pi B_\lambda$ and hence it is compact. It is almost evident that A is a completely regular space with $\operatorname{ind} A=0$. Thus the theorem is proved.

Theorem 2. *A non-empty metric space R is always the image of a metric space A with* $\dim A=0$, *under the open continuous mapping f such that $f^{-1}(x)$ is compact for every point x of R.*

Proof. Since a metric space is always paracompact by A. H. Stone [4, Corollary 1], there exists a sequence $\mathfrak{U}_i=\{U_\alpha; \alpha\in A_i\}$, $i=1, 2,\cdots$, of locally finite open coverings of R such that the diameter of each element of $\mathfrak{U}_i$ is less than $1/i$. Let A be the aggregate of points $a=(\alpha_i; i=1, 2,\cdots)$ of the product space $\Pi\{A_i; i=1, 2,\cdots\}$, where A_i are topological spaces with the discrete topology, such that $\frown\{U_{\alpha_i}; i=1, 2,\cdots\}\neq\phi$. Let $f(a)=\frown\{U_{\pi_i(a)}; i=1, 2,\cdots\}$, where $\pi_i: A\to A_i$, $i=1, 2,\cdots$, are the projections. Then by the same argument as in the proof of Theorem 1 f becomes an open continuous mapping of A onto R such that $f^{-1}(x)$ is compact for every point x of R. Moreover A is a metric space with $\dim A=0$ by Katětov [1, Theorem 3.7] or Morita [2, Theorem 10.2]. Thus the theorem is proved.

When R is a separable metric space we can impose the following additional condition upon a sequence $\mathfrak{U}_i$, $i=1, 2, \cdots$, in the above proof: Every $\mathfrak{U}_i$ consists of countable elements. In this case A in the above proof is separable. Hence we have the following.

Theorem 3. *A non-empty separable metric space R is always the image of a separable metric space A with $\dim A=0$, under the open continuous mapping f such that $f^{-1}(x)$ is compact for every point x of R.*

References

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