84. *Harmonic Condition in the Theory of Unified Field*

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ABSTRACT. The linearised equations of a modified version of Einstein's field theory have their aspect deeply changed if one assumes that the cosmological constant, called here fundamental, may take very high values with respect to the antisymmetrical field of the electromagnetism.

One justifies then, physically, the choice of a generalised condition of harmonicity which leads through the gravitational equations of the theory to the definition of a mass to charge ratio.

The application of a variational principle on the fields $L^{\alpha}_{\beta\gamma}$, $\mathfrak{G}^{\mu\nu}$, S_μ, a, $\mathfrak{A}^\mu$ through the following modified*) lagrangian:

$$(1)\qquad \mathfrak{L}\equiv\mathfrak{G}^{\mu\nu}[W_{\mu\nu}-m\Pi^{-}_{\underline{\mu\nu}}]+2\mathfrak{A}^\mu L_\mu+a(\mathfrak{G}^{\mu\nu}S_\mu S_\nu-2\sqrt{-g}\,\alpha^2)$$

leads if one assumes that the antisymmetric part of the fundamental tensor $\varphi_{\mu\nu}$ is small enough $\simeq\varepsilon$ and after that the equations in series of powers of ε have been expanded, to the approximate equations [4]:

$$(2)\qquad \square\varphi_{\mu\nu}=K_0\Pi^{-}_{\mu\nu}-G^{\cdot\tau\rho}_{\cdot\mu\nu}\varphi_{\tau\rho}+G_{\tau\nu}\varphi^{\cdot\tau}_{\cdot\mu}-G_{\tau\mu}\varphi^{\cdot\tau\nu}+2a\alpha^2\varphi_{\mu\nu}+2KS^{\cdot\lambda}\nabla_\lambda\varphi_{\mu\nu}$$
$$+\frac{K}{6}(\nabla_\mu\varphi_{\nu\tau}-\nabla_{\cdot}\varphi_{\mu\tau})S^{\cdot\tau}-\frac{2K}{3}[\nabla_\nu(\varphi_{\sigma\mu}S^{\cdot\sigma})-\nabla_\mu(\varphi_{\sigma\nu}S^\sigma)]$$

$$(3)\qquad G_{\mu\nu}+aS_\mu S_\nu-a\alpha^2\gamma_{\mu\nu}=-\frac{1}{6}[\nabla_\mu(KS_\nu)+\nabla_\nu(KS_\mu)]+\frac{K^2}{2}S_\mu S_\nu$$
$$+\frac{1}{2}\nabla^{\cdot\rho}(\varphi^{\cdot\sigma}_{\cdot\nu}(\nabla_\mu\varphi_{\rho\sigma}-\nabla_\sigma\varphi_{\mu\rho}+\nabla_\rho\varphi_{\sigma\mu})+\text{sym. for }\mu,\nu]$$
$$-\frac{1}{6}\nabla^{\cdot\rho}[KS^{\cdot\lambda}(\varphi_{\mu\lambda}\gamma_{\nu\rho}+\varphi_{\nu\lambda}\gamma_{\mu\rho}-3\gamma_{\mu\nu}\varphi_{\rho\lambda})]$$
$$-\Big(-\frac{1}{2}\varphi_{\mu\rho\cdot}^{\ \ \ \lambda}+\nabla^{\cdot\lambda}\varphi_{\mu\rho}\Big)\Big(-\frac{1}{2}\varphi_{\lambda\nu\cdot}^{\ \ \ \rho}+\nabla^{\cdot\rho}\varphi_{\lambda\nu}\Big)$$

(where $\square$ represents $\nabla^{\cdot\lambda}\nabla_\lambda$, ∇ the covariant derivative in the γ metric: $\gamma_{\mu\nu}\equiv g_{\underline{\mu\nu}}$ symmetric, the dotted indices being raised or lowered by γ $K_0=a/3m+2m$; $K=a/m$; $G_{\mu\nu}=G^{\rho}_{\cdot\mu\nu\rho}$ (Ricci))

*) This change has been brought [1] in order to elude particular limits of the theory and to give the opportunity to arrive at $\partial_\lambda\mathfrak{G}^{\underset{\vee}{\mu\tau}}\neq0$; $L^{\alpha}_{\beta\gamma}$ is the affine connexion of the Cartan's generalised space, to which we assign the condition $L_\alpha\equiv L^{\rho}_{\underset{\vee}{\alpha\rho}}=\frac{1}{2}(L^{\rho}_{\alpha\rho}-L^{\rho}_{\rho\alpha})$ through the multiplicator $\mathfrak{A}^\mu\cdot\mathfrak{G}^{\mu\nu}=\sqrt{-g}g^{\mu\nu}$ fundamental density, S_μ phenomenological vector holding the normalisation condition in $2\alpha^2$ assigned by the multiplicator a. We will assume in what follows, in order to get a formal simplification, a practically constant.

$$\mathfrak{G}^{\mu\nu}S_\mu S_\nu = 2\sqrt{-g}\,\alpha^2. \tag{4}$$

The gravitational equation (3) appears under a form similar to that of the general relativity

$$G_{\mu\nu} - \Lambda\gamma_{\mu\nu} = -\chi T_{\mu\nu} \qquad \text{general relativity} \tag{5}$$

$$G_{\mu\nu} - a\alpha^2\gamma_{\mu\nu} = -\left[aS_\mu S_\nu + \frac{1}{6}(\nabla_\mu KS_\nu + \nabla_\nu KS_\mu) + \cdots\right] \qquad \text{unified field}$$

where the Ricci' tensor $G_{\mu\nu}$ and the cosmological term are counter-balanced by the stress-energy-momentum tensor of the right hand side.

Let us introduce now a very important hypothesis joining the ideas of several authors [5, 6, 7, 8]: the cosmological constant, hereafter called fundamental, may take very high values with respect to the antisymmetric field $\varphi_{\mu\nu}$ which belongs to electromagnetism.

The fundamental constant. Usually one considers in general relativity [9] the constant Λ of (5) to be very small (inverse of square of the universe radius). Let us then immediately make a remark: Einstein in particular has shown that this constant is found to be proportional to the material density considered as being uniformly spread on a large scale [10]. Therefore by choosing Λ small, it is natural to find oneself again in the cosmological case. The immediate extension therefore consists in supposing this fundamental constant, localised or not, very high and this with respect to a field of the theory taken as a reference field. We shall get within these conditions large scalar curvatures, localised or not, hence large energetical concentrations.

But the elementary particles correspond effectively to huge densities: 10^{15} g/cm^3 for the nucleons.

It seems therefore now natural enough to propose this hypothesis of a highly valued fundamental constant in unified field theory. Equations then take a very new aspect:

$$aS_\mu S_\nu - a\alpha^2\gamma_{\mu\nu} + G_{\mu\nu} + \varepsilon\cdots = 0 \tag{6}$$

$$2m\Pi^-_{\mu\nu} + 2a\alpha^2\varphi_{\mu\nu} - (G^{\tau\rho}_{\cdot\cdot\mu\nu}\varphi_{\tau\rho} - G_{\tau\nu}\varphi^\tau_{\cdot\mu} + G_{\tau\mu}\varphi^\tau_\nu) + \varepsilon\cdots = 0. \tag{7}$$

Note here that, contrary to general relativity, the generalised curvature does not proceed on $G_{\mu\nu}$ alone. So it is necessary to dissociate in the contribution to the generalised scalar curvature $g^{\mu\nu}W_{\mu\nu}$ the electromagnetic part from the gravitation one, which may be with the masses very small. In another way we shall have to assume the potential $\gamma_{\mu\nu}$ to be quasi-galilean and admit the compatibility of this assumption with the one on the fundamental constant.

Follow then the general relativity methods and apply the divergence operator to the Einstein tensor built from (7): we choose [1, 2 p. 97, 5, 11, 12] the metric

(8) $$\lambda^{\alpha\beta}=\sqrt{\frac{h}{g}}\,h^{\alpha\beta} \qquad h^{\alpha\beta}=g^{\underline{\alpha\beta}} \text{ symmetric} \quad h, g \text{ determinant of } h_{\mu\nu},\ g_{\mu\nu}$$

and we take:

(9) $$\partial_\sigma[\sqrt{-\lambda}\,\lambda^{\sigma\theta}(G_{\mu\nu}-\lambda_{\mu\nu}G_{\alpha\beta}\lambda^{\alpha\beta})].$$

For a slowly varying field of gravitation potential: $\gamma_{\mu\nu}=\delta_{\mu\nu}$ constant $+\eta_{\mu\nu}$ small function, the identical cancellation is easily pointed out to the first order of $\eta_{\mu\nu}$. The left-hand side of (9) then gives, thanks to particular relations between determinants elements (see [2], φ determinant of $\varphi_{\mu\nu}$):

(10) $$\sqrt{-g}\,h^{\sigma\rho}S_\rho\partial_\sigma S_\mu-\alpha^2\gamma_{\mu\rho}\partial_\sigma(\sqrt{-g}\,h^{\sigma\rho})-\alpha^2\Big(\sqrt{-g}\,h^{\sigma\rho}\partial_\rho\gamma_{\mu\sigma}-\partial_\mu\frac{\varphi-\gamma}{\sqrt{-g}}\Big)=0$$

that we must interpret now. This point can be reached only in assigning a coordinate choice.

The harmonic condition. This is our second assumption. In the generalised Cartan spaces the covariance is spared in a frame which distorsions by degrees tell the manifold properties. We therefore think that the physical phenomena are absorbed in these changes and that the only way to make them appear out the equations is to fix a state of the frame family by some particular supplementary conditions, for example to the density $\mathfrak{G}^{\mu\nu}$ otherwise to the affine connexion. The choice is critical because it must reflect our phenomena overlooking.

This attitude is similar to Fock's [13] and it is why it has seemed to us logical then to assign the harmonicity condition.

But, one is able to give an other point of view in order to justify this choice. Given a coordinates transformation:

(11) $$x'^\mu=x^\mu+\xi^\mu$$

in assigning the preceding condition, one limits ξ^μ and confines the allowed transformations, then the Lorentz's group appears [13, 14]. But, within these conditions we shall see that the particles movement also appears.

Therefore, it seems that, in staying outside the particles, only way to discern their Lorentz movement, one confines the general transformation group.

In the same way, if we stay inside—without the harmonicity condition—keeping the general transformation group, one must reach the "internal movement".

However, this aspect of the problem will have to be precised later.

Therefore, from our metric choice (8), we shall assign to draw out the trajectories, the condition:

(12) $$\partial_\sigma(\sqrt{-g}\,h^{\mu\sigma})=\partial_\sigma(\sqrt{-\lambda}\,\lambda^{\mu\sigma})=0.$$

Using the relations between the determinants given in [2], assuming $\varphi_{\mu\nu}$ small and keeping approximations up to the second order, it comes:

$$\frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}\,h^{\mu\sigma})\simeq j'^{\mu}_{\cdot}-\partial_\sigma\tau^{\mu\sigma}=0 \tag{13}$$

where

$$j'_\mu=\gamma^{\rho\sigma}\partial_\sigma\gamma_{\rho\mu}-\frac{1}{\sqrt{-g}}\partial_\mu\frac{\gamma}{\sqrt{-g}}$$

$$\tau^{\beta\sigma}=\frac{1}{4}\gamma^{\beta\sigma}\varphi^{\rho\lambda}_{\cdot\cdot}\varphi_{\rho\lambda}-\varphi^{\beta\rho}_{\cdot\cdot}\varphi^{\sigma}_{\cdot\rho}.$$

Let us introduce (12) and (13) into (10). To the chosen approximation it stays:

$$S^\sigma\partial_\sigma S_\mu=\alpha^2\partial_\sigma\tau_\mu^{\cdot\sigma} \tag{14}$$

and in this way, thanks to (4) taken as

$$S_\mu=\sqrt{2}\,\alpha u_\mu, \qquad h^{\sigma\lambda}u_\sigma u_\lambda=1$$

and to the first term of (6) yielding a rotational structure for the field $\varphi_{\mu\nu}$, this equation (14) brings us the opportunity to compare with the general relativity results:

$$u^\lambda\partial_\lambda u_\mu=J^\lambda\varphi_{\mu\lambda} \qquad \text{general relativity}$$

$$u^\lambda\partial_\lambda u_\mu=\frac{K\alpha}{\sqrt{2}}u^\lambda\varphi_{\mu\lambda} \qquad \text{unified field.}$$

We may define then a charge to mass ratio through the scalar $K\alpha$.

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