81. On the Bend of Continuous Plane Curves

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In the theory of functions of a real variable there is a beautiful theorem of importance due to S. Banach (cf. Saks [2], p. 280):

THEOREM OF BANACH. *Let $F(x)$ be a continuous real function on a linear closed interval I and let $s(y)$ denote for each real number y the (finite or infinite) number of the points of I at which F assumes the value y. Then the function $s(y)$ is B-measurable and its integral over the real line coincides with* $\mathrm{W}(F; I)$, *i.e. the absolute variation of F over I.*

The condition that I is a *closed* interval is not essential for the validity of the assertion. With slight modifications in the proof we have the same result even when I is an arbitrary interval of real numbers; only we then interpret $\mathrm{W}(F; I)$ as the weak variation of F over I (defind on p. 221 of Saks [2]).

We established in our paper [1] certain basic properties of a geometric quantity called curve bend. It is the object of the present note to obtain an analogue of the Banach theorem for the bend of a plane curve determined by an equation of the form $y=F(x)$, where again continuity is the sole condition that we impose upon the function F. Though our theorem is similar to that of Banach in enunciation, the proof turns out far more complicated in our case. We presuppose complete knowledge of [1] on the part of the reader. The precise statement of our theorem reads as follows:

THEOREM. *Let us define $p(\theta)=\langle\cos\theta, \sin\theta\rangle$ for the points θ of the interval $K=[-\pi/2, \pi/2]$. Given on a linear interval I_0 (of any type) a continuous real function $F(x)$, let $f(\theta)$ denote for each $\theta\in K$ the number (finite or infinite) of the points of I_0 at which the unit-vector $p(\theta)$ is a derived direction* (see [1] §42) *for the curve φ defined on I_0 by $\varphi(x)=\langle x, F(x)\rangle$. Then $f(\theta)$ is a B-measurable function on K and its integral over K coincides with $\Omega(\varphi)$, i.e. the bend of φ.*

All the notations of this theorem will be retained throughout the rest of the present note. Since the function $p(\theta)$ is continuous and biunique, so is also its inverse function p^{-1}, which maps the semicircle $p[K]$ onto the interval K. It is immediately seen further that if θ_1 and θ_2 are any pair of points of K, then the angle $p(\theta_1)\diamond p(\theta_2)$ is equal to $|\theta_1-\theta_2|$ (see [1] §21).

This being so, let us begin by proving the following analogue of the Banach theorem for curve length.

LEMMA 1. *Suppose that $\psi(t)$ is a continuous curve defined on a linear interval J_0 and situated in the semicircle $p[K]$. For each point θ of K, let $g(\theta)$ denote the number of the points of J_0 at which ψ assumes the value $p(\theta)$. Then $g(\theta)$ is a B-measurable function on K and its integral over K coincides with $\Lambda(\psi)$, i.e. the spheric length of ψ.*

REMARK. Since ψ is continuous, $\Lambda(\psi)$ is equal to the ordinary length of ψ on account of [1] §76. For our purpose, however, it is more convenient to consider $\Lambda(\psi)$.

PROOF. For each point t of J_0 let $G(t)$ stand for the inverse image of the point $\psi(t)$ under the mapping p. Then G is a continuous function on J_0 with values belonging to K, and we find at once that $W(G; J_0)=\Lambda(\psi; J_0)$. Furthermore it is obvious that $g(\theta)$ coincides for each $\theta \in K$ with the number of the points of J_0 at which the function G assumes the value θ. The assertion follows now at once from the theorem of Banach.

DEFINITIONS. Let $T(x)$ be a real-valued function on I_0 and let c be any point of I_0. (As we have alreadys observed, every notation of our theorem will keep its meaning in the course of our argument, so that I_0 always denotes a linear interval of any type.) An infinite sequence $J_1, J_2, \cdots$ of closed intervals will as usual be termed to *tend to* c iff (i.e. if and only if) every J_n contains c and further $|J_n| \to 0$ as $n \to +\infty$. Now let ξ be an extended real number, i.e. a real number or $\pm\infty$. We shall say that ξ is a *derived number* of the function T at the point c iff there exists in I_0 an infinite sequence $J_1, J_2, \cdots$ of closed intervals tending to c and such that $T(J_n)/|J_n| \to \xi$ as $n \to +\infty$. It is evident that *this is the case when and only when $p(\mathrm{Tan}^{-1}\xi)$ is a derived direction at c of the curve τ defined for $x \in I_0$ by $\tau(x)=\langle x, T(x)\rangle$.* (The symbol Tan^{-1} denotes the principal value of the inverse tangent belonging to the interval $K=[-\pi/2, \pi/2]$, where and subsequently we understand $+\infty$ by $\tan(\pi/2)$ and $-\infty$ by $\tan(-\pi/2)$). Finally, given a triple ξ_0, ξ_1, ξ_2 of extended real numbers, ξ_0 will be termed to *lie between ξ_1 and ξ_2* iff we have one or both of the relations $\xi_1 \leqq \xi_0 \leqq \xi_2$ and $\xi_1 \geqq \xi_0 \geqq \xi_2$.

LEMMA 2. *Let c be an interior point of the interval I_0 and suppose that the function F of the theorem possesses at c unilateral derivatives on the right and left, denoted by α and β respectively. In order that an extended real number ξ be then a derived number of F at c it is necessary and sufficient that ξ should lie between α and β.*

PROOF. We shall confine ourselves to the case $\alpha<\beta$. Consider in I_0 an arbitrary pair of closed intervals P and Q of which c is the right-hand and the left-hand extremity respectively. If these intervals are sufficiently short, then $F(P)/|P|>F(Q)/|Q|$ and so

$$F(P)/|P|>F(P\smile Q)/|P\smile Q|>F(Q)/|Q|.$$

The necessity of the condition is an immediate consequence of this.

Consider next a fixed value ξ_0 such that $\alpha<\xi_0<\beta$. We proceed to associate with each natural number n a closed interval $[a_n, b_n]\subset I_0$ and a real number λ_n so as to fulfil the following two requirements: (i) $a_n<c<b_n$ and $b_n-a_n<n^{-1}$; (ii) if we write $H_n(x)=F(x)-\xi_0 x-\lambda_n$ for every point x of I_0, then $H_n(a_n)<0$, $H_n(b_n)<0$, and $H_n(c)>0$. For this purpose we need only choose firstly $[a_n, b_n]\subset I_0$ sufficiently short in order to secure condition (i) and to fulfil further the two inequalities $F(c)-F(a_n)>\xi_0(c-a_n)$ and $F(b_n)-F(c)<\xi_0(b_n-c)$. Indeed $F(c)-\xi_0 c$ then exceeds $A=\max[F(a_n)-\xi_0 a_n,\ F(b_n)-\xi_0 b_n]$, and so there exists a λ_n such that $F(c)-\xi_0 c>\lambda_n>A$. But the last relation is plainly equivalent to condition (ii).

On account of the intermediate value theorem there then exist two points u_n and v_n such that $a_n<u_n<c<v_n<b_n$ and $H_n(u_n)=H_n(v_n)=0$. If we now write for brevity $J_n=[u_n, v_n]$, then $J_1, J_2,\cdots$ constitute a sequence of closed intervals lying in I_0 and tending to the point c, and we have $F(J_n)/|J_n|=\xi_0$ for every n. This shows that ξ_0 is a derived number of F at c. As, moreover, both α and β are obviously derived numbers of F at c, we conclude that the condition of our lemma is sufficient.

LEMMA 3. *Given in I_0 a triple of points $a<c<b$ and given a pair of real coefficients A and B, let us write $H(x)=F(x)-Ax-B$ for each point x of I_0. If $H(a)\leqq 0$, $H(b)\leqq 0$, and $H(c)\geqq 0$, then the open interval (a, b) contains a point at which A is a derived number for the function F.*

PROOF. 1) Consider first the case $H(c)>0$. We may suppose $H(a)=H(b)=0$. For, if $H(a)<0$ for instance, there is by the intermediate value theorem a point a' fulfilling both $a<a'<c$ and $H(a')=0$, and we need only replace the point a by a'. Now the function H attains its maximum on $[a, b]$ at some point c' of $[a, b]$, where we must have $a<c'<b$ since the assumption $H(c)>0$ implies $H(c')>0$. It clearly suffices to show that zero is a derived number of H at this point c'. For this purpose we may assume c' to be a point of strict maximum for H, and the result then holds by the intermediate value theorem.

2) It remains to deal with the case $H(c)=0$. By what has just been proved in part 1) we may suppose $H(x)$ nonpositive everywhere in the interval $[a, b]$. Then H attains at the point c its maximum on $[a, b]$, and the assertion easily follows by arguing as at the end of part 1).

LEMMA 4. *Given in* I_0 *a triple of points* $a<c<b$, *let us write* $P=[a, c]$ *and* $Q=[c, b]$. *Then each real number* ξ *which lies between the two quotients* $F(P)/|P|$ *and* $F(Q)/|Q|$ *is a derived number of* F *at some point of the open interval* (a, b).

PROOF. We may suppose $F(P)/|P|\geqq\xi\geqq F(Q)/|Q|$ without loss of generality. Write $B=F(c)-\xi c$ and define $H(x)=F(x)-\xi x-B$ for each point $x\in I_0$. Then $H(c)=0$ and further $H(a)=|P|\xi-F(P)\leqq 0$, $H(b)=F(Q)-|Q|\xi\leqq 0$. Lemma 3 shows now at once the truth of the assertion.

LEMMA 5. *For each subset* E *of* I_0 *let* $M(E)$ *denote the set of the points* θ *of* K *such that* $p(\theta)$ *is a derived direction of the curve* φ *at some point of* E, *or equivalently, such that* $\tan\theta$ *is a derived number of the function* F *at some point of* E. *Then the set* $M(E)$ *is convex whenever* E *is a one-point set or an open interval.*

REMARK. As is almost evident, a nonvoid set of real numbers is convex iff it is either a one-point set or an interval. We shall retain the symbol $M(E)$ throughout the rest of this note.

PROOF. We have to ascertain that every closed interval $[\theta_1, \theta_2]$ with extremities belonging to $M(E)$ is necessarily contained in $M(E)$. Suppose $\theta_1<\theta_0<\theta_2$ for this purpose and write $\xi_i=\tan\theta_i (i=0, 1, 2)$ for short, so that $\xi_1<\xi_0<\xi_2$.

1) Consider first the case where E consists of a single point c. Since both ξ_1 and ξ_2 are derived numbers of F at c, there exists in I_0, for each positive number ε, a pair of closed intervals $P_i=[a_i, b_i]$ $(i=1, 2)$ which contain the point c, have lengths $<\varepsilon$, and fulfil the relation $F(P_1)/|P_1|<\xi_0<F(P_2)/|P_2|$. We now attach to each point t of $[0, 1]$ a closed interval $J_t=[a_1(1-t)+a_2t, b_1(1-t)+b_2t]$, so that $c\in J_t\subset I_0$ and $|J_t|=|P_1|(1-t)+|P_2|t<\varepsilon$. Then the function ξ defined by $\xi(t)=F(J_t)/|J_t|$ for $t\in[0, 1]$ is continuous and we clearly have $\xi(i-1)=F(P_i)/|P_i|$ for both $i=1$ and 2. Consequently there is in $(0, 1)$ a point t_0 for which $\xi(t_0)=\xi_0$. Since ε is arbitrary, it follows that ξ_0 is a derived number of F at c, or equivalently, that $\theta_0\in M(E)$. This completes the proof for $E=\{c\}$.

2) We pass on to the case where E is an open interval. By definition of $M(E)$ there is in E a distinct pair of points c_1 and c_2 such that ξ_i is a derived number of F at c_i for $i=1, 2$. We then can choose in E a disjoint pair of closed intervals I_1 and I_2 containing the points c_1 and c_2 respectively and satisfying $F(I_1)/|I_1|<\xi_0<F(I_2)/|I_2|$. Let I_3 be the closed interval that abuts both I_1 and I_2, so that ξ_0 lies either between $F(I_1)/|I_1|$ and $F(I_3)/|I_3|$ or between $F(I_2)/|I_2|$ and $F(I_3)/|I_3|$. It follows from Lemma 4 that ξ_0 is a derived number of F at some point of E, or what amounts to the same thing, that $\theta_0\in M(E)$, Q.E.D.

LEMMA 6. *The function $f(\theta)$ is B-measurable and $\Omega(\varphi)$ does not exceed twice the integral of $f(\theta)$ over K.*

PROOF. Given any natural number n, let us decompose the interval I_0 into a disjoint sequence Δ_n (finite or infinite) each of whose elements J is either a one-point set or an open interval with length smaller than n^{-1}. Let further $S_E(\theta)$ denote for each subset E of I_0 the characteristic function of the set $M(E)$. Since $S_J(\theta)$ is a B-measurable function of θ for every element J of Δ_n by Lemma 5, so must also be the sum of $S_J(\theta)$ for all J. On the other hand, writing $f_n(\theta)$ for this sum, we easily verify that $f_n(\theta)\to f(\theta)$ at each point θ of the interval K as $n\to+\infty$. This proves $f(\theta)$ to be a B-measurable function on K.

Let us now insert in I_0 an arbitrary sequence $x_0<x_1<\cdots<x_{n+1}$ of $n+2$ points, n being any natural number. To shorten our notations we put $Q_i=[x_{i-1}, x_i]$ and $R_j=(x_{j-1}, x_{j+1})$, where and below the index i ranges over $1,\cdots, n+1$ and j over $1,\cdots, n$. Then every $M(R_j)$ is a convex set on account of Lemma 5. The function $S_E(\theta)$ defined above will conveniently be written $S(\theta; E)$ in what follows. Noting that then $S(\theta; R_1)+\cdots+S(\theta; R_n)\leqq 2f(\theta)$ for every $\theta\in K$ as is easily seen, we deduce at once, with the help of Lemma 4, that

$$\sum_j[\varphi(Q_j)\diamond\varphi(Q_{j+1})]\leqq\sum_j|M(R_j)|=\int_K[\sum_j S(\theta; R_j)]d\theta\leqq 2\int_K f(\theta)d\theta.$$

This implies the inequality of the assertion, since $\Omega(\varphi)$ is the supremum of the leftmost sum in the above relation for all choices of the sequence $x_0,\cdots, x_{n+1}$.

PROOF OF THE THEOREM. On account of Lemma 6 we may assume $\Omega(\varphi)$ finite. For any interval $I\subset I_0$ endless on the right [or on the left] (see [1] §72), the restriction of the curve φ to I must be C^R on I [or C^L on I] in virtue of [1] §80. It thus follows easily, in view of [1] §32, that we need only consider the case where the interval I_0 is endless and where therefore φ is C^{RL} on I_0. We then have $\Omega(\varphi)=\Lambda(\varphi^R)$ by the theorem of [1] §96. Consequently our theorem will be established if we show $\Lambda(\varphi^R)=A$, where and subsequently A denotes for brevity the integral of $f(\theta)$ over K.

Now we find by [1] §83 that the curve φ^R is right-hand continuous and that $\varphi^R(x-)=\varphi^L(x)$ everywhere in I_0. Hence the equality $\varphi^R(x)=\varphi^L(x)$ is equivalent for each $x\in I_0$ to continuity of φ^R at x. It follows from the proof of [1] §78 that, at each point u of continuity of φ^R, the curve φ has a tangent direction $\hat{\varphi}(u)$ equal to $\varphi^R(u)=\varphi^L(u)$, so that $\hat{\varphi}(u)$ is a unique derived direction of φ at u. If, therefore, φ^R is a continuous curve in particular, the relation $\Lambda(\varphi^R)=A$ is a direct consequence of Lemma 1.

Let us pass on to the case in which the set N of the points of discontinuity for φ^R is nonvoid. Since N is countable on account of rectifiability of φ^R, there exists by [1] §94 a continuous non-decreasing function $W(t)$, defined on an endless interval J_0 and mapping J_0 onto I_0, and such that the inverse image $W^{-1}(x)$ of a point x of I_0 is a non-degenerate set, and hence a closed interval, when and only when $x \in N$. Such an interval will as usual be called *interval of constancy* (of the function W). We proceed to construct a continuous mapping $\psi(t)$ of J_0 into the semicircle $p[K]$ as follows. Writing $t^* = W(t)$ for short for any point t of I_0, we distinguish two cases according as $t^* \in N$ or not. In the latter case we put simply $\psi(t) = \varphi^R(t^*)$. In the former case, on the other hand, write $[a, b]$ for the interval of constancy that contains the point t, and let θ_1 and θ_2 denote the inverse images, under the mapping p, of the distinct points $\varphi^L(t^*)$ and $\varphi^R(t^*)$ respectively. Let us then set $\psi(t) = p(\theta_1 + (\theta_2 - \theta_1)\lambda)$, where λ is determined by the equation $t = a + (b-a)\lambda$.

Thus defined on J_0 the spheric curve ψ is easily seen to be continuous. Further, ψ is biunique on each interval J of constancy of W and fulfils $\Lambda(\psi; J) = \varphi^L(t^*) \diamond \varphi^R(t^*)$ for any point t of J. In view of the last relation we find without difficulty that $\Lambda(\varphi^R) = \Lambda(\psi)$. Our task thus reduces itself to proving $\Lambda(\psi) = A$. Now let $g(\theta)$ denote for $\theta \in K$ the number of the points of J_0 at which ψ assumes the value $p(\theta)$. Then $\Lambda(\psi)$ equals the integral of $g(\theta)$ over K in virtue of Lemma 1. The proof will therefore be complete if we verify that $g(\theta) = f(\theta)$ identically. But this is an easy consequence of Lemma 2 by what we have stated in the above about the curve ψ.

References

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[2] S. Saks: Theory of the Integral, Warszawa-Lwów (1937).