47. Algebraic Equaton, whose Roots lie in a Unit Circle or in a Half-plane.

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I. *Algebraic equations, whose roots lie in a unit circle.*

1. In this paper $\bar{a}$ means the conjugate complex of a. Let

$$f(x)=a_0+a_1x+\dots+a_nx^n,\quad f^*(x)=x^n\bar{f}\left(\frac{1}{x}\right)=\bar{a}_n+\bar{a}_{n-1}x+\dots+\bar{a}_0x^n, \tag{1}$$

$$A=\begin{pmatrix} a_0, & a_1, & a_2, \dots, & a_{n-1} \\ 0, & a_0, & a_1, \dots, & a_{n-2} \\ 0, & 0, & a_0, \dots, & a_{n-3} \\ \dots\dots & & & \\ 0, & 0, & 0, \dots, & a_0 \end{pmatrix},\qquad \bar{A}'=\begin{pmatrix} \bar{a}_0, & 0, & 0, & \dots, 0 \\ \bar{a}_1, & \bar{a}\ , & 0, & \dots, 0 \\ \bar{a}_2, & \bar{a}_1, & \bar{a}_0, & \dots, 0 \\ \dots\dots & & & \\ \bar{a}_{n-1}, & \bar{a}_{n-2}, & \bar{a}_{n-3}, & \dots, \bar{a}_0 \end{pmatrix},$$

$$B=\begin{pmatrix} \bar{a}_n, & \bar{a}_{n-1}, & \bar{a}_{n-2}, \dots, & \bar{a}_1 \\ 0, & \bar{a}_n, & \bar{a}_{n-1}, \dots, & \bar{a}_2 \\ 0, & 0, & \bar{a}_n, \dots, & \bar{a}_3 \\ \dots\dots & & & \\ 0, & 0, & 0, \dots, & \bar{a}_n \end{pmatrix},\qquad \bar{B}'=\begin{pmatrix} a_n, & 0, & 0, & \dots, 0 \\ a_{n-1}, & a_n, & 0, & \dots, 0 \\ a_{n-1}, & a_{n-1}, & a_n, & \dots, 0 \\ \dots\dots & & & \\ a_1, & a_2, & a_3, & \dots, a \end{pmatrix},$$

$$\mathfrak{H}=\bar{B}'B-\bar{A}'A=(\gamma_{ik}),\ |\mathfrak{H}|=\text{det.}\ (\gamma_{ik}),$$

$$\mathfrak{H}(x)=\sum_0^{n-1}\gamma_{ik}x_i\,\bar{x}_k,\ (\gamma_{ki}=\bar{\gamma}_{ik}), \tag{2}$$

$$\delta_\nu=\left|\begin{array}{llll|llll} a_n, & 0, & \dots, & 0, & a_0, & a_1, & \dots, & a_{\nu-1} \\ a_{n-1}, & a_n, & \dots, & 0, & 0, & a_0, & \dots, & a_{\nu-2} \\ \dots\dots & & & & \dots\dots & & & \\ a_{n-\nu+1}, & a_{n-\nu+2}, & \dots, & a_n, & 0, & 0, & \dots, & a_0 \\ \hline \bar{a}_0, & 0, & \dots, & 0, & \bar{a}_n, & \bar{a}_{n-1}, & \dots, & \bar{a}_{n-\nu+1} \\ \bar{a}_1, & \bar{a}_0, & \dots, & 0, & 0, & \bar{a}_n, & \dots, & \bar{a}_{n-\nu+2} \\ & & & & \dots\dots & & & \\ \bar{a}_{\nu-1}, & \bar{a}_{\nu-2}, & \dots, & \bar{a}_0, & 0, & 0, & \dots, & \bar{a}_n \end{array}\right|,\ (\nu=1, 2,\dots, n). \tag{3}$$

We denote the determinant of a matrix A by $|A|$ and its ν-th section by A_ν, which is a matrix formed with elements of A lying in the first ν rows and first ν columns. Then as Schur[1] proved,

$$\delta_\nu = \begin{vmatrix} \overline{B}'_\nu, & A_\nu \\ \overline{A}'_\nu, & B_\nu \end{vmatrix} = |\overline{B}'_\nu B_\nu - \overline{A}'_\nu A_\nu| = |(\overline{B}'B - \overline{A}'A)_\nu|, \text{ so that } \delta_n = |\mathfrak{H}|. \quad (4)$$

Then the following theorems hold:

Theorem 1 (I. Schur).[2] *The necessary and sufficient condition, that all roots of $f(x)=0$ lie in a unit circle $|x|<1$ is that the Hermitian form $\mathfrak{H}(x)$ is positive definite, or $\delta_1>0$, $\delta_2>0, \dots, \delta_n>0$.*

As Cohn proved,[3] $\delta_n = R(f, f^*)$, where $R(f, f^*)$ is the resultant of $f(x)$ and $f^*(x)$, so that $\mathfrak{H}(x)$ is of rank n, when and only when $f(x)$ and $f^*(x)$ have no common factor.

Theorem 2 (Cohn).[3] *If $f(x)$, $f^*(x)$ have no common factor, then $\mathfrak{H}(x)$ is of rank n and when reduced to the normal form:*

$$\mathfrak{H}(x) = |y_1|^2 + |y_2| + \dots + |y_\pi|^2 - |z_1|^2 - |z_\nu|^2 - \dots - |z_\nu|^2,$$

π is the number of roots of $f(x)=0$ in $|x|<1$ and ν is that in $|x|>1$.

We will give a simple proof of Theorem 2.

2. *Proof of Theorem 2.*

We assume that $f(x)$ and $f^*(x)$ have no common factor, so that $f(x)=0$ has no root on $|x|=1$ and $\mathfrak{H}(x)$ is of rank n. Let $f(x)=0$ have π roots in $|x|<1$ and ν roots in $|x|>1$, then $f^*(x)=0$ has ν roots in $|x|<1$ and π roots in $|x|>1$.

We assume that $a_0 \neq 0$, $a_n \neq 0$, $a_0 + \bar{a}_n \neq 0$ and $f(x)+f^*(x)=0$ has no double roots. Let $f(x)+f^*(x)=0$ have p roots (ϵ_k) in $|x|<1$ and q roots (η_k) on $|x|=1$, then it has p roots $\left(\frac{1}{\bar{\epsilon}_k}\right)$ is $|x|>1$, so that $2p+q=n$. Since $f^*(x)-f(x)$, $f^*(x)+f(x)$ have no common factor, we have in the neighbourhood of x,

$$\frac{f^*(x)-f(x)}{f^*(x)+f(x)} = \frac{c}{2} + \sum_{k=1}^{p} \left(\frac{r_k}{2} \cdot \frac{\epsilon_k + x}{\epsilon_k - x} + \frac{r'_k}{2} \cdot \frac{1+\bar{\epsilon}_k x}{1-\bar{\epsilon}_k x} \right) + \sum_{k=1}^{q} \frac{r_k^0}{2} \cdot \frac{\eta_k + x}{\eta_k - x}$$

$$= \frac{c+c_0}{2} + \sum_{n=1}^{\infty} c_n x^n, \ (2p+q=n), \quad (5)$$

where $|\epsilon_k|<1$, $|\eta_k|=1$.

In the both sides of (5), we put $\frac{1}{x}$ in place of x and take the conjugate

1) I. Schur: Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. Crelle, **147** (1917).

2) I. Schur: Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind (Fortsezung). Crelle **148** (1918).

3) A. Cohn: Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise. Math. Zeits. **14** (1922).

complex of the quantities involved, then the left hand side changes its sign. From this, we conclude that $\bar{c}=-c$, $r'_k=\bar{r}_k$, $\bar{r}^0_k=r^0_k$, so that c is purely imaginary and r^0_k are real. Hence

$$c_n=\sum_{k=1}^{p}(r_k\epsilon_k^{-n}+\bar{r}_k\bar{\epsilon}_k^{n})+\sum_{k=1}^{q}r_k^0\eta_k^{-n},\ (c_0=\text{real}).\quad (n=0,1,2,\dots) \quad (6)$$

We consider a Hermitian form:

$$H(x)=\sum_{0}^{n-1}c_{k-i}x_i\bar{x}_k,\ (c_{-k}=\bar{c}_k),\ H=\begin{pmatrix} c_0, & c_1, & \dots, & c_{n-1} \\ \bar{c}_1, & c_0, & \dots, & c_{n-2} \\ \dots\dots & & & \\ \bar{c}_{n-1}, & \bar{c}_{n-2}, & \dots, & c_0 \end{pmatrix} \quad (7)$$

Then, as Schur[4] proved, we have easily

$$(\bar{B}'+\bar{A}')H(B+A)=2(\bar{B}'B-\bar{A}'A)=2\mathfrak{H}, \quad (8)$$

since $|B+A|=(a_0+\bar{a}_n)^n\neq 0$, two Hermitian forms $H(x)$ and $\mathfrak{H}(x)$ are equivalent, so that $H(x)$ is of rank n.

Now

$$H(x)=\sum_{k=1}^{p}H_k(x)+\sum_{\kappa=1}^{q}H_k^0(x), \quad (9)$$

where $H_k(x)$ is a Hermitian form formed with $(r_k\epsilon_k^{-\nu}+\bar{r}_k\bar{\epsilon}_k^{\nu})$ and $H_k^0(x)$ is that with $(r_k^0\eta_k^{-\nu})$, $(\nu=0, 1, 2,\dots, n-1)$.

Since

$$\begin{vmatrix} r_k+\bar{r}_k, & r_k\epsilon_k^{-1}+\bar{r}_k\bar{\epsilon}_k \\ \bar{r}_k\bar{\epsilon}_k^{-1}+r_k\epsilon_k, & r_k+\bar{r}_k \end{vmatrix}=|r_k|^2\left(2-|\epsilon_k|^2-\frac{1}{|\epsilon_k|^2}\right)<0,$$

$H_k(x_0, x_1, 0, \dots, 0)$ is an indefinite form and since as easily be seen, $H_k(x)$ is of rank 2, $H_k(x)$ can be reduced to the normal form: $H_k(x)=|y_k|^2-|z_k|^2$.

Since $H_k^0(x)=r_k^0|x_0+x_1\bar{\eta}_k+\dots+x_{n-1}\bar{\eta}_k^{n-1}|^2$, if we denote the numbers of positive and negative r_k^0 by α, β respectively, then $H(x)$ can be reduced to the nomal from:

$$H(x)=|y_1|^2+|y_2|^2+\dots+|y_{p+\alpha}|^2-|z_1|^2-|z_2|^2-\dots-|z_{p+\beta}|^2,$$
$$(2p+\alpha+\beta=n). \quad (10)$$

We will prove that $p+\alpha=\pi$, $p+\beta=\nu$.

Let $x=x(\lambda)$ be the root of $f(x)+\lambda f^*(x)=0$, so that $\eta_k=x(1)$, $f(\eta_k)+f^*(\eta_k)=0$. Then from $(f'(x)+\lambda f^{*\prime}(x))dx+f^*(x)d\lambda=0$ and (5), we have

$$dx=\frac{-f^*(\eta_k)d\lambda}{f'^*(\eta_k)+f'(\eta_k)}=\frac{f(\eta_k)-f^*(\eta_k)}{2(f'^*(\eta_k)+f'(\eta_k))}d\lambda=\frac{r_k^0}{2}\eta_k d\lambda, \quad \text{or}$$

$$dx=x\frac{r_k^0}{2}d\lambda \quad \text{at} \quad x=\eta_k,\ \lambda=1. \quad (11)$$

4) I. Schur : l.c. (1).

Hence $f(x)+\lambda f^*(x)=0$ has a root in $|x|<1$ in a neighbourhood of η_k, if $r_k^0>0$, $1-\delta<\lambda<1$, or $r_k^0<0$, $1<\lambda<1+\delta$, when δ is small. Since $f(x)+f^*(x)=0$ has p roots in $|x|<1$, we conclude that

$$f(x)+\lambda f^*(x)=0 \text{ has } p+\alpha \text{ roots in } |x|<1, \text{ if } 1-\delta<\lambda<1 \text{ and}$$
$$p+\beta \text{ roots in } |x|<1, \text{ if } 1<\lambda<1+\delta. \quad (12)$$

On the other hand, since $f(x)$ has π roots in $|x|<1$ and $f^*(x)$ has ν roots in $|x|<1$ and $|f(x)|=|f^*(x)|$ on $|x|=1$, we have by Rouché's theorem,

$$f(x)+\lambda f^*(x)=0 \text{ has } \pi \text{ roots in } |x|<1, \text{ if } 0<\lambda<1 \text{ and}$$
$$\nu \text{ roots in } |x|<1, \text{ if } 1<\lambda. \quad (13)$$

From (12), (13), we have $p+\alpha=\pi$, $p+\beta=\nu$, so that from (10),

$$H(x)=|y_1|^2+|y_2|^2+\dots+|y_\pi|^2-|z_1|^2-|z_2|^2\dots-|z_\nu|^2,$$
$$(\pi+\nu=n). \quad (14)$$

Since $H(x)$ is of rank n. n linear forms $y_1, y_2,\dots y_\pi, z_1, z_2,\dots, z_\nu$ of $x_0, x_1,\dots x_{n-1}$ are linearly independent.

Since as remarked above, $H(x)$ and $\mathfrak{H}(x)$ are equivalent, $\mathfrak{H}(x)$ can be reduced to the normal form of the form (14). Hence Cohn's theorem is proved under the assumption, that $a_0\neq 0$, $a_n\neq 0$, $a_0+\bar{a}_n\neq 0$ and $f(x)+f^*(x)=0$ has no double roots. But as Cohn remarked this restriction can be removed as follows. We change the coefficients of $f(x)$ slightly, so that this condition is satisfied. Let $\mathfrak{H}'(x)$ be the corresponding Hermitian form, then $\mathfrak{H}'(x)$ can be reduced to the normal form of the form (14). Since $|\mathfrak{H}|\neq 0$, we see easily that the numbers π and ν remain unchanged, when the variations of coefficients are sufficiently small, so that $\mathfrak{H}(x)$ can be reduced to the normal form of the form (14), which proves Theorem 2.

3. Remark.

If $f(x)$ and $f^*(x)$ have the greatest common factor $g(x)$ of degree $m(>0)$, then $\mathfrak{H}(x)$ becomes of rank $n-m$. We assume, for brevity, that $g(x)=0$ has no double roots. Let $g(x)=0$ have p roots (ϵ_k) in $|x|<1$ and q roots (η_k) on $|x|=1$, then it has p roots $\left(\frac{1}{\bar{\epsilon}_k}\right)$ in $|z|>1$, so that

$$\frac{g'(x)}{g(x)}=\sum_{k=1}^{p}\left(\frac{1}{x-\epsilon_k}+\frac{1}{x-\frac{1}{\bar{\epsilon}_k}}\right)+\sum_{k=1}^{q}\frac{1}{x-\eta_k}=\sum_{n=0}^{\infty}\frac{c_n}{x^{n+1}}, \ (2p+q=m),$$

$c_n=\sum_{k=1}^{p}(\epsilon_k^n+\bar{\epsilon}_k^{-n})+\sum_{k=1}^{q}\eta_k^n=s_n$(sum of the n-th power of the roots), where $|\epsilon_k|<1$, $|\eta_k|=1$. We form a Hermitian form:

$$H(x)=\sum_{0}^{m-1}c_{k-i}x_i\bar{x}_k, \ (c_{-k}=\bar{c}_k) \quad (15)$$

and as before we decompose $H(x)$ into the form: $H(x)=\sum_{k=1}^{p}H_k(x)+\sum_{k=1}^{q}H_k^0(x)$,

where $H_k(x)$ is a Hermitian form formed with $(\epsilon_k^\nu+\bar{\epsilon}_k^{-\nu})$ and $H_k^0(x)$ is that with $(\eta_k^\nu)(\nu=0, 1, 2,\dots, m-1)$. Then as before, we see easily that $H(x)$ can be reduced to the normal form:

$$H(x)=|y_1|^2+|y_2|^2+\dots+|y_{p+q}|^2-|z_1|^2-|z_2|^2-\dots-|z_p|^2, \quad (16)$$

where p is the number of roots of $g(x)=0$ in $|x|<1$ and q is that on $|x|=1$. If $g(x)=0$ has multiple roots, then p and q are the numbers of distinct roots, multiple roots being counted once. From (16), we have:

The necessary and sufficient condition, that all roots of $g(x)=0$ lie on $|x|=1$ is that $H(x)$ is positive definite.

II. *Algebraic equations, whose roots lie in a half-plane.*

Let

$$f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n,\ g(x)=b_1x^{n-1}+b_2x^{n-2}+\dots+b_n$$
$$(a_0\neq 0,\ b_1\neq 0) \quad (17)$$

$$\frac{g(x)}{f(x)}=\sum_{n=0}^{\infty}\frac{c_n}{x^{n+1}},$$

$$R_k=\left|\begin{matrix} a_0, & a_1, & \dots\dots, & a_{2k-2} \\ 0, & a_0, & \dots\dots, & a_{2k-3} \\ \dots & & & \\ 0, & 0, & \dots a_0,\dots, & a_k \\ \hline b_1, & b_2, & \dots\dots, & b_{2k-1} \\ 0, & b_1, & \dots\dots, & b_{2k-2} \\ \dots & & & \\ 0, & 0, & \dots, b_1, \dots, & b_k \end{matrix}\right| \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} k-1 \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} k \end{matrix}, \quad C_0^{(k)}=\left|\begin{matrix} c_0, & c_1, & \dots, & c_k \\ c_1, & c_2, & \dots, & c_{k+1} \\ \dots & & & \\ c_k, & c_{k+1}, & \dots, & c_{2k} \end{matrix}\right|,$$

where $a_\nu=0$, $b_\nu=0$, if $\nu>n$. Then as well known,[5)]

$$R_k=(-1)^{\frac{k(k-1)}{2}}a_0^{2k-1}C_0^{(k-1)}, \text{ so that}$$

$$R_n=R(f, g)=(-1)^{\frac{n(n-1)}{2}}a_0^{2n-1}C_0^{(n-1)}, \quad (18)$$

where $R(f, g)$ is the resultant of f and g. Hence if $f(x)$ and $g(x)$ have no common factor, then $C_0^{(n-1)}\neq 0$. Let

$$f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n,\ \bar{f}(x)=\bar{a}_0x^n+\bar{a}_1x^{n-1}+\dots+\bar{a}_n,$$
$$(\bar{a}_0=a_0>0). \quad (19)$$

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5) 藤原松三郎：代數學第一卷 462 頁, Netto, Crelle **116** (1896), Netto, Algebra I. p. 86.

We assume that $f(x)$ and $\bar{f}(x)$ have no common factor. Then $f(x)=0$ has no real roots. Let $f(x)=0$ has π roots in $\Im x<0$ and ν roots in $\Im x>0$. Then $\bar{f}(x)=0$ has ν roots in $\Im x<0$ and π roots in $\Im x>0$. We as- sume that $f(x)+\bar{f}(x)=0$ has no double roots. Let $f(x)+\bar{f}(x)=0$ have p roots (ϵ_k) in $\Im x>0$ and q roots (η_k) on the real axis, then it has p roots $(\bar{\epsilon}_k)$ in $\Im x<0$. Since $f(x)+\bar{f}(x)$, $f(x)-\bar{f}(x)$ have no common factor, we have

$$\frac{f(x)-\bar{f}(x)}{i(f(x)+\bar{f}(x))}=\sum_{k=1}^{p}\left(\frac{r_k}{x-\epsilon_k}+\frac{r'_k}{x-\bar{\epsilon}_k}\right)+\sum_{k=1}^{q}\frac{r_k^0}{x-\eta_k}$$

$$=\sum_{n=0}^{\infty}\frac{c_n}{x^{n+1}},\ (2p+q=n), \tag{20}$$

where $\Im\epsilon_k>0$ and η_k are real.

If we take the conjugate complex of the quantities involeved in (20), then the left hand side is unchanged. From this, we conclude that $r'_k=\bar{r}_k$, $\bar{r}_k^0=r_k^0$, so that r_k^0 are real. Hence

$$c_n=\sum_{k=1}^{p}(r_k\epsilon_k^n+\bar{r}_k\bar{\epsilon}_k^n)+\sum_{k=1}^{q}r_k^0\eta_k^n, \tag{21}$$

so that c_n are real. We form a real quadratic form:

$$\mathfrak{H}(x)=\sum_{0}^{n-1}c_{i+k}x_ix_k,\ |\mathfrak{H}|=C_0^{(n-1)}, \tag{22}$$

Since $f(x)-\bar{f}(x)$, $f(x)+\bar{f}(x)$ have no common factor, $C_0^{(n-1)}\neq 0$ as remarked in the begining of the proof, so that $\mathfrak{H}(x)$ is of rank n

Let as before, $\mathfrak{H}(x)=\sum_{k=1}^{p}H_k(x)+\sum_{k=1}^{q}H_k^0(x)$, where $H_k(x)$ is a quadratic form formed with $(r_k\epsilon_k^\nu+\bar{r}_k\bar{\epsilon}_k^\nu)$ and $H_k^0(x)$ is that with $(r_k^0\eta_k^\nu)$, $(\nu=0,1,2,\dots,n-1)$. Since

$$\begin{vmatrix} \bar{r}_k+r_k, & r_k\epsilon_k+\bar{r}_k\bar{\epsilon}_k \\ r_k\epsilon_k+\bar{r}_k\bar{\epsilon}_k, & r_k\epsilon_k^2+\bar{r}_k\bar{\epsilon}_k^2 \end{vmatrix}=|r_k|^2(\epsilon_k-\bar{\epsilon}_k)^2<0,$$

$H_k(x, x_1, 0\dots0)$ is an indefinite form. Since as easily be seen, $H_k(x)$ is of rank 2, $H_k(x)$ can be reduced to the normal form: $H_k(x)=y_k^2-z_k^2$.

Since $H_k^0(x)=r_k^0(x_0+x_1\eta_k+\dots+x_{n-1}\eta_k^{n-1})^2$, if we denote the numbers of positive and negative r_k^0 by α, β respectively, then $\mathfrak{H}(x)$ can be reduced to the normal form:

$$\mathfrak{H}(x)=y_1^2+y_2^2+\dots+y_{p+\alpha}^2-z_1^2-z_2^2-\dots-z_{p+\beta}^2,$$
$$(2p+\alpha+\beta=n). \tag{23}$$

We will prove that $p+\alpha=\pi$, $p+\beta=\nu$.

Let $x=x(\lambda)$ be the root of $f(x)+\lambda f(x)=0$, so that $\eta_k=x(1)$, $f(\eta_k)+\bar{f}(\eta_k)=0$. Then from (20), we have as before, at $x=\eta_k$, $\lambda=1$,

$$dx=\frac{-f(\eta_k)}{f'(\eta_k)+\bar{f}'(\eta_k)}d\lambda=\frac{f(\eta_k)-\bar{f}(\eta_k)}{2(f'(\eta_k)+\bar{f}'(\eta_k))}d\lambda=i\frac{r_k^0}{2}d\lambda, \quad \text{or}$$

$$dx=i\frac{r_k^0}{2}d\lambda \quad \text{at } x=\eta_k,\ \lambda=1. \tag{24}$$

Hence $f(x)+\lambda\bar{f}(x)=0$ has a root in $\Im x>0$ in a neighbourhood of η_k, if $r_k^0>0, 1<\lambda<1+\delta$, or $r_k^0<0, 1-\delta<\lambda<1$, when δ is small. Since $\bar{f}(x)+f(x)=0$ has p roots in $\Im x>0$, we conclude that

$$f(x)+\lambda\bar{f}(x)=0 \text{ has } p+\alpha \text{ roots in } \Im x>0, \text{ if } 1<\lambda<1+\delta \text{ and}$$
$$p+\beta \text{ roots in } \Im x>0, \text{ if } 1-\delta<\lambda<1. \tag{25}$$

On the other hand, since $\dfrac{f(x)}{(x-i)^n}$ and $\dfrac{\bar{f}(x)}{(x-i)^n}$ are regular in $\Im x\geqq 0$, $x=\infty$ being included and $\left|\dfrac{f(x)}{(x-i)^n}\right|=\left|\dfrac{\bar{f}(x)}{(x-i)^n}\right|$ on the real axis, if we map $\Im x>0$ on $|z|<1$ conformally and apply Rouché's theorem on $\dfrac{f(x)}{(x-i)^n}+\lambda\dfrac{\bar{f}(x)}{(x-i)^n}=0$, then we see that

$$f(x)+\lambda\bar{f}(x)=0 \text{ has } \pi \text{ roots in } \Im x>0, \text{ if } 1<\lambda \text{ and}$$
$$\nu \text{ roots in } \Im x>0, \text{ if } 0<\lambda<1. \tag{26}$$

From (25), (26), we have $p+\alpha=\pi$, $p+\beta=\nu$, so that from (23),

$$\mathfrak{H}(x)=y_1^2+y_2^2+\dots+y_\pi^2-z_1^2-z_2^2-\dots-z_\nu^2,\ (\pi+\nu=n). \tag{27}$$

Since $\mathfrak{H}(x)$ is of rank n, n linear forms $y_1, \dots, y_\pi, z_1, \dots, z_\nu$ are linearly independent.

We assumed that $\bar{f}(x)+f(x)=0$ has no double roots. If this condition is not satisfied, then we change the coefficients of $f(x)$ slightly and conclude as before that $\mathfrak{H}(x)$ can be reduced to the normal form of the form (27).

Hence we have:

Theorem 3. Let $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$, $\bar{f}(x)=\bar{a}_0x+\bar{a}_1x^{n-1}+\dots+\bar{a}_n$, $(a_0>0,\ a_n=\alpha_n+i\beta_n)$. *We assume that $f(x)$ and $\bar{f}(x)$ have no common factor. Let*

$$\frac{f(x)-\bar{f}(x)}{i(f(x)+\bar{f}(x))}=\frac{\beta_1x^{n-1}+\beta_2x^{n-2}+\dots+\beta_n}{a_0x^n+a_1x^{n-1}+\dots+a_n}=\sum_{n=0}^{\infty}\frac{c_n}{x^{n+1}},\ \mathfrak{H}(x)=\sum_{0}^{n-1}c_{i+k}x_ix_k.$$

Then, $\mathfrak{H}(x)$ is of rank n and when reduced to the normal form;

$$\mathfrak{H}(x)=y_1^2+y_2^2+\dots+y_\pi^2-z_1^2-z_2^2-\dots-z_\nu^2,\ (\pi+\nu=n),$$

π is the number of roots of $f(x)=0$ in $\Im x<0$ and ν is that in $\Im x>0$.

Since by (18),

$$R_k=\left|\begin{array}{l} a_0,\ a_1,\cdots\cdots,\ \ a_{2k-2} \\ 0,\ a_0,\cdots\cdots,\ \ a_{2k-3} \\ \cdots\cdots\cdots \\ 0,\ 0,\ \cdots a_0,\cdots,a_k \\ \hline \beta_1,\ \beta_2,\cdots\cdots,\ \beta_{2k-1} \\ 0,\ \beta_1,\cdots\cdots,\ \beta_{2k-2} \\ \cdots\cdots\cdots \\ 0,\ 0,\cdots\beta_1,\cdots,\beta_k \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\}k-1 \\ \left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\}k \end{array} =(-1)^{\frac{k(k-1)}{2}}a_0^{2k-1}C_0^{(k-1)},$$

we have:

Theorem 4. The necessary and sufficient condition, that all roots of $f(x)=0$ *lie in* $\mathfrak{I}x<0$ *is that* $\mathfrak{H}(x)$ *is positive definite, or* $(-1)^{\frac{k(k-1)}{2}}R_k>0$, $(k=1, 2,\dots, n)$.

From Theorem 3, we have

Theorem 5. Let $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$, $(a_0>0,\ a_n=\alpha_n+i\beta_n)$. *We assume that* $f(-ix)$ *and* $\bar{f}(-ix)$ *have no common factor. Let*

$$F(x)=\frac{f(-ix)}{(-i)^n},$$

$$\frac{F(x)-\bar{F}(x)}{i(F(x)+\bar{F}(x))}=\frac{\alpha_1x_{n-1}+\beta_2x^{n-2}-\alpha_3x^{n-3}-\beta_4x^{n-4}+\alpha_5x^{n-5}+\beta_6x^{n-6}-\dots}{a_0x^n+\beta_1x^{n-1}-\alpha_2x^{n-2}-\beta_3x^{n-3}+\alpha_4x^{n-4}+\beta_5x^{n-5}-\dots}$$

$$=\sum_{n=0}^{\infty}\frac{c_n}{x^{n+1}},\quad \mathfrak{H}(x)=\sum_0^{n-1}c_{i+k}x_ix_k.$$

Then, $\mathfrak{H}(x)$ *is of rank n and when reduced to the normal form:*

$$\mathfrak{H}(x)=y_1^2+y_2^2+\dots+y_\pi^2-z_1^2-z_2^2-\dots-z_\nu^2,\ (\pi+\nu=n),$$

π *is the number of roots of* $f(x)=0$ *in* $\mathfrak{R}x<0$ *and* ν *is that in* $\mathfrak{R}x>0$.

From Theorem 5, we have the following extension of Hurwitz's theorem,[6] who assumed that all a_n are real.

Theorem 6. The necessary and sufficient condition, that all roots of $f(x)=0$ *lie in* $\mathfrak{R}x<0$ *is that* $\mathfrak{H}(x)$ *is positive definite.*

Analogous theorems as Theorem 3, 4, 5, 6 were proved by M. Fujiwara[7] by means of Bézoutians.

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6) A. Hurwitz: Über die Bedingung, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt. Math. Ann. **46** (1895).

7) M. Fujiwara: Über die algebraischen Gleichungeu, deren Wurzeln in eimem Kreise oder in einer Halbebene liegen. Math. Zeits. **24** (1926).