42. Note on the theory of conformal representation by meromorphic functions. I.

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§ 1. *Preliminaries.*

We consider, in general, on the one hand a family of analytic functions

$$(1.1)\qquad \{g(\zeta)\},\quad g(\infty)=\infty,\ g'(\infty)=1,$$

defined on $|\zeta|>1$ and normalized at $\zeta=\infty$, as is here explicitly written, and on the other a family of analytic functions

$$(1.2)\qquad \{f(z)\},\quad f(0)=0,\qquad f'(0)=1,$$

defined in $|z|<1$ and normalized at $z=0$. We can then establish a ono-to-one correspondence between them by the relations

$$(1.3)\quad \zeta z=1,\quad g(\zeta)f(z)=1\quad \text{i.e.}\quad g(\zeta)=f(\zeta^{-1})^{-1},\ f(z)=g(z^{-1})^{-1};$$

the corresponding functions $g(\zeta)$ and $f(z)$ behave, furtheremore, both analytic and *schlicht* (univalent) at the same time in their respective domains of definition, the case to which we shall confine ourselves in the following lines. Under these circumstances any properties of the one family imply at once the corresponding ones of the other. As a matter of fact, it is especially remarkable that the so-called Bieberbach's area theorem concerning the former family has paved a way also to a systematic development of the theory of the latter.

But the considerations on the latter are often confined to the sub-family, consisting of regular functions only, that is to say, consisting of only those functions $f(z)$ which correspond, by (1.3), to special functions $g(\zeta)$ vanishing nowhere on $|\zeta|>1$. Various properties of this sub-family have been hitherto, indeed partly by an essential utilization of the supplementary restriction in question, i.e. the regularity, investigated in full detail. When the family (1.1) is, however, supposed to be merely schlicht, we should rather consider the schlicht and generally *meromorphic* family (1.2) itself which correspond, by the relations (1.3), exactly to the whole family (1.1). The results on the family just ranged have been, though often of importance and very useful, established hitherto comparatively few.

Even if we assume that the family (1.2) of schlicht functions are meromorphic, each member $f(z)$ has, as a matter of course, at most only one pole of the first order in the basic circle $|z|<1$. We shall however consider here, from the above-stated point of view, the whole family (1.2) of schlicht and normalized functions, inclusive of ones having a pole in the unit-circle, and note some interesting consequences, connected with this family, of well-known theorems in the theory[1] of the family (1.1). The corresponding problems for the case of doubly-connected basic domain will also be explicitly treated in later papers.

§ 2. *Problems to be considered.*

Let $f(z)$ denote preliminarily, a function, regular analytic and schlicht in $|z|<1$, which is moreover normalized at the origin i.e. such that $f(0)=0$ and $f'(0)=1$. Then its beginning Taylor coefficients $a_2=f''(0)/2$ and $a_3=f'''(0)/6$ are always subject to the restrictions

$$(2.1) \qquad |a_2|\leqq 2 \qquad \text{and} \qquad |a_3|\leqq 3$$

by the theorems of Bieberbach[2] and Löwner[3] respectively. Accordingly if, for instance, only one of them does not hold, the function must possess necessarily a pole, of course as previously noticed, of the first order in the interior of the unit-circle. We may therefore propose in general the question how the position of the pole, z_∞ say, if exists at all and its residue

$$(2.2) \qquad A=\lim_{z\to z_\infty}(z-z_\infty)f(z)=\left[-\frac{f(z)^2}{f'(z)}\right]_{z=z_\infty}$$

must be restricted, when such coefficient a_2 or a_3 is preassigned. This is obviously almost equivalent with an analogous one relating to the position of the zero-point

$$(2.3) \qquad \zeta_0=\frac{1}{z_\infty}$$

of $g(\zeta)$ and its differential coefficient at this point, viz.

$$(2.4) \qquad g'(\zeta_0)=-\frac{z_\infty^2}{A},$$

$g(\zeta)$ denoting here, of course, the function which corresponds to $f(z)$ by the relations (1.3).

It is a matter of common knowledge that the so-called Koebe-Bieberbach's

1) Various results on this discipline obtained up to the present are, together with the detailed list of literatures including those cited in the following, collected in the author's recently published book: Conformal representation, I. (Japanese) 1944.

2) L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. Sitzungsber. preuß. Akad. Wiss. Berlin (1916), 940–955.

3) K. Löwner, Untersuchungen über schlichte konforme Abbildnngen des Einheitskreises, I. Math. Annalen **89** (1923), 103–121.

one fourth theorem[4] takes a fundamental position in the whole theory of regular schlicht fnnctions in $|z|<1$. But, in the case of meromorphic functions, a famous theorem of Montel,[5] plays an analogous rôle, on which further discussions have been done by Bieberbach,[6] Fenchel[7] and the others.[8] In connection with this last-mentioned theorem, we shall take here a glance also at the ranges covered by image-domains which arise from $|z|<1$ by any transformation $w=f(z)$ in question.

§ 3. *Ranges of the transformed domains.*

We consider in the first place, with a view to obtaining a result of Fenchel afresh but more briefly, a function $g(\zeta)$ of the family (1.1) whose Laurent expansion about the point at infinity is evidently of the form

$$g(\zeta)=f(\zeta^{-1})^{-1}=\zeta+b_0+\frac{b_1}{\zeta}+\dots \tag{3.1}$$

with beginning coefficients

$$b_0=-a_2, \quad b_1=a_2^2-a_3, \dots . \tag{3.2}$$

Hence, by means of a well-known theorem of Löwner[9] which states

$$|g(\zeta)-b_0|\leqq|\zeta|+\frac{1}{|\zeta|} \qquad (|\zeta|>1) \tag{3.3}$$

we have at once an inequality

$$\left|\frac{1}{f(z)}+a_2\right|\leqq\frac{1}{|z|}+|z| \qquad (|z|<1). \tag{3.4}$$

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4) P. Koebe, Über die Uniformisierung beliebiger analytischer Kurven. Nachr. Ges. Wiss. Göttingen (1907), 191–210: Über die Uniformisierung der algebraischen Kurven, I. Math. Annalen **69** (1909), 145–224; Abhandlungen zur Theorie der konformen Abbildung, III. Der allgemeine Fundamentalsatz der konformen Abbildung nebst einer Anwendung auf die konforme Abbildung der Oberfläche einer körperlichen Ecke. Crelles Journal **147** (1917), 67–104; etc. L. Bieberbach, Über einige Extremalprobleme im Gebiete der konformen Abbildung. Math. Annalen **77** (1916), 153–172. Cf., also G. Faber, Neuer Beweis eines Koebe-Bieberbachschen Satzes über konforme Abbildung. Sitzungsber. Bayer. Akad. Wiss. München (1916), 39–42.

5) P. Montel, Sur les domaines formés par les points représentant les valeurs d'une fonction analytique. Ann. Sci. École Norm. Sup. (3) **46** (1929), 1–23.

6) L. Bieberbach, Über schlichte Abbildung des Einheitskreises durch meromorphe Funktionen. Sitzungsber. preuß. Akad. Berlin (1929), 620–624; Über schlichte Abbildung des Einheitskreises durch meromorphe Funktionen, II. ibid. (1937), 3–9.

7) W. Fenchel, Bemerkungen über die im Einheitskreises meromorphen schlichten Funkionen. Sitzungsber. preuß. Akad. Berlin (1931), 431–436.

8) Cf., for example, E. Rengel, Über einige Schlitztheoreme der konformen Abbildung. Schriften d. math. Sem. u. Inst. f. angew. Math. d. Univ. Berlin **1** (1932), 141–162.

9) K. Löwner, Über Extremalsätze bei der konformen Abbildung des Äußeren des Einheitskreises. Math. Zeitschr. **3** (1919), 65–77.

Thus the distortion theorem of Fenchel[10] is an immediate consequence of the last inequality, that is, we have at any case

$$(3.5) \qquad |f(z)| \geqq \frac{|z|}{1+|a_2||z|+|z|^2}$$

for $|z|<1$, and further, if $|a_2|>2$,

$$(3.6) \qquad |f(z)| \leqq \frac{|z|}{-1+|a_2||z|-|z|^2}$$

for $h(|a_2|)<|z|<1$, where $h=h(a)$ $(a>2)$, $0<h<1$, denotes the unique positive root of the quadratic equation

$$h^2-ah+1=0,$$

i.e.

$$(3.7) \qquad h(a)=\frac{a}{2}-\sqrt{\frac{a^2}{4}-1}.$$

The former inequality (3.5) was, however, already noticed by Gronwall,[11] by using the method just adopted here. According to Fenchel, we can then conclude from (3.6) that the position of pole is limited by the relation

$$(3.8) \qquad |z_\infty| \leqq h(|a_2|)=\frac{|a_2|}{2}-\sqrt{\frac{|a_2|^2}{4}-1},$$

provided $|a_2|>2$. But this result follows more directly also from (3.4). In fact, we have, since $f(z_\infty)=\infty$, the inequality

$$(3.9) \qquad |a_2| \leqq \frac{1}{|z_\infty|}+|z_\infty|,$$

which implies at once the proposition (3.8).

An opposite estimation for the distance $|z_\infty|$ of the pole from the origin is lacking in the above result. Therefore, we attempt now, as some complement, to deduce such an estimation. We make here, in order to obtain first a lower bound depending only on $|\zeta|$ for the quantity $|g(\zeta)|$, use of the Bieberbach's area theorem,[12] viz.

$$(3.10) \qquad \sum_{n=1}^{\infty} n|b_n|^2 \leqq 1.$$

It follows then, by using the Schwarz's inequality together, that

$$\sum_{n=1}^{\infty}\frac{|b_n|}{|\zeta|^n} \leqq \sqrt{\sum_{n=1}^{\infty} n|b_n|^2 \sum_{n=1}^{\infty}\frac{1}{n|\zeta|^{2n}}} \leqq \sqrt{\lg\frac{|\zeta|^2}{|\zeta|^2-1}}$$

and hence

10) W. Fenchel, loc. cit. 8).

11) T. H. Gronwall, On the distortion in conformal mapping when the second cofficient in the mapping function has an assigned value. Proc. Nat. Acad. Sci. 6 (1920), 300-302.

12) L. Bieberbach, loc. cit. 2); G. Faber, loc. cit. 4).

$$|g(\zeta)| \geqq |\zeta| - |b_0| - \sqrt{\lg \frac{|\zeta|^2}{|\zeta|^2 - 1}}.$$

The equality sign appears here, as is easily verified, save at $\zeta = \infty$ not at all. Thus, denoting by $k = k(a)$ $(a \geqq 0)$, $0 < k < 1$, the unique positive root of the transcendental equation

$$\frac{1}{k} = a + \sqrt{\lg \frac{1}{1-k^2}},$$

we have an inequality of disortion, i.e.

$$|f(z)| < \frac{|z|}{1 - |a_2||z| - |z|\sqrt{\lg \frac{1}{1-|z|^2}}}$$

for $0 < |z| \leqq k$ *and at the same time an estimation*

(3.11) $$|z_\infty| > k(|a_2|)$$

which is of a desired form.[13]

But the result just derived is rough enough, while the limit given in (3.8) is surely attained by, and only by, the functions of the form

(3.12) $$f(z) = \frac{z}{1 - a_2 z + \epsilon^2 z^2} \qquad (|a_2| > 2),$$

where ϵ is a complex number with absolute value unity and with the same argument as a_2, i.e. $\epsilon = a_2/|a_2|$.

The function (3.12) gives, as is easily seen, the conformal representation of $|z| < 1$ onto the whole plane cut along a segment joining the two points $-\frac{\varepsilon}{|a_2| \pm 2}$.

In order to obtain his distortion formulae (3.5) and (3.6), Fenchel has first proved the following theorem:

The image of $|z| < 1$ by the transformation $w = f(z)$ contains always $|w| < \frac{1}{|a_2| + 2}$, and contains further $|w| > \frac{1}{|a_2| - 2}$ also provided $|a_2| > 2$.

After we have, however, obtained the distortion formulae (2.5) and (3.6) previously, the theorem just stated follows conversely from them almost immediately.

Now, with regard to the Laurent coefficients about the origin of regular schlicht and normalized functions, the precise limitation is, as has been already noticed in (2.1), known also for $|a_3|$ besides $|a_2|$. In connection with this circumstance, we show here that an analogous theorem can also be obtained, if $|a_3|$ is preassigned:

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13) Some other results belonging to the same category as (3.8) or (3.11) will be discussed in the later paragraph § 6.

Theorem 1. *The image of* $|z|<1$ *by the transformation* $w=f(z)$ *contains always* $|w|<\dfrac{1}{\sqrt{|a_3|+1}+2}$, *and contains further* $|w|>\dfrac{1}{\sqrt{|a_3|+1}-2}$ *also provided* $|a_3|>3$.

We state, before proof, a preparatory remark on the relation of this theorem to the Fenchel's. As we have, by means of the inequality $|b_1|\leqq 1$ followed from (3.10) and of the second relation in (3.2), always the inequality

$$|a_3|+1\geqq|a_2|^2, \tag{3.13}$$

the first half of the theorem is evidently contained in that of the Fenchel's. But, provided that $|a_2|>2$, we have, by (3.13), necessarily $|a_3|>3$ and furtheremore

$$\frac{1}{|a_2|-2}\geqq\frac{1}{\sqrt{|a_3|+1}-2}.$$

The latter half of theorem gives therefore generally a better limitation than that of Fenchel.

Now, to prove the theorem in question, we introduce a function defined by

$$F(z)=\frac{cf(z)}{c-f(z)}=z+\left(a_2+\frac{1}{c}\right)z^2+\left(a_3+\frac{2a_2}{c}+\frac{1}{c^2}\right)z^3+\cdots,$$

where c denotes any boundary point of the image of $|z|<1$ by the mapping $w=f(z)$. The function $F(z)$ being regular and schlicht in $|z|<1$, it follows at once from the generally valid estimations in (2.1) that

$$\left|a_2+\frac{1}{c}\right|\leqq 2 \quad\text{and}\quad \left|a_3+\frac{2a_2}{c}+\frac{1}{c^2}\right|\leqq 3. \tag{3.14}$$

Hence, the former inequality, together with (3.13), implies

$$|c|\geqq\frac{1}{|a_2|+2}\geqq\frac{1}{\sqrt{|a_3|+1}+2}, \tag{3.15}$$

while we obtain from the latter, by taking the former too into account, in order the inequalities

$$a\geqq|a_3|-\frac{2}{|c|}\left|a_2+\frac{1}{c}\right|-\frac{1}{|c|^2},$$

$$|a_3|+1\leqq 4+\frac{4}{|c|}+\frac{1}{|c|^2}=\left(2+\frac{1}{|c|}\right)^2,$$

and finally provided $|a_3|>3$,

$$|c|\leqq\frac{1}{\sqrt{|a_3|+1}-2}. \tag{3.16}$$

Both estimations (3.15) and (3.16) are nothing but what was to be proved.

We complete next the theorem just proved by the following supplementary theorem.

Theorem 2. *The limits mentioned in the preceding theorem are both precise.*

To prove this, we observe the inequalities (3.15) and (3.16). In a converse manner as the derivations show, the either equality can appear, by virtue of (3.2), only if there exists a complex number ε such that

$$a_2^2-a_3=\epsilon^2=e^{2i\arg a_2}=e^{i\arg a_3}, \qquad \epsilon=\frac{a_2}{|a_2|},$$

and moreover if the mapping function is of the form

$$f(\zeta^{-1})^{-1}=\zeta-a_2+\epsilon^2\zeta^{-1},$$

$$(3.17) \qquad f(z)=\frac{z}{1-a_2z+\epsilon^2z^2}=\frac{z}{1-\epsilon\sqrt{|a_3|+1}\,z+\epsilon^2z^2},$$

which coïncides just with that given in (3.12). The functions of this form map $|z|<1$, as is already noticed, onto the whole plane cut along a slit between the two points

$$-\frac{\bar{\varepsilon}}{|a_2|\pm 2}=-\frac{\bar{\varepsilon}}{\sqrt{|a_3|+1}\pm 2}$$

Hence, these and only these functions attain indeed the limits.

The already named Montel-Bieberbach's theorem, which can be regarded as a consequence of Fenchel's result, can also be derived very briefly from our theorem. In fact, if $|a_3|\leqq 4$, then

$$\frac{1}{\sqrt{|a_3|+1}+2}\geqq\frac{1}{\sqrt{5}+2}=\sqrt{5}-2;$$

and if $|a_3|\geqq 4$, then

$$\frac{1}{\sqrt{|a_3|+1}-2}\leqq\frac{1}{\sqrt{5}-2}=\sqrt{5}+2.$$

Hence, the image of $|z|<1$ by the mapping $w=f(z)$ covers always either $|w|<\sqrt{5}-2$ or $|w|>\sqrt{5}+2$, and the functions (3.17) with $|a_3|=4$ and those alone are extreme for both bounds; this is just the content of the Montel-Bieberbach's theorem.

Now, we shall further remark that *an another source of this theorem can be found in a theorem due to Landau,*[14] which states that, if a function

$$\psi(z)=\frac{1}{z}+\sum_{n=0}^{\infty}\gamma_n z^n$$

is schlicht and non-vanishing in $|z|<1$, the inequality

14) E. Landau, Der Picard-Schottkysche Satz und die Blochsche Konstante. Sitzungsber. preuß. Akad. Wiss. Berlin (1926), 467–474.

$$|\gamma_0| \leqq 2V\left(\frac{1+\theta}{4}\right)$$

is valid for any quantity θ with $|\gamma_1| \leqq \theta < 1$, where the function $V=V(N)$ is defined by the parametric equations

$$N=\left(\nu+\frac{1}{2}\right)e^{-\nu}, \quad V=(\nu+1)e^{-\nu}.$$

In fact, applying this theorem to the function

$$\frac{1}{F(z)}-\frac{1}{c}=\frac{1}{z}+\left(a_2+\frac{1}{c}\right)+(a_2^2-a_3)z+\ldots$$

with any bauudary point c, we obtain at once

$$\left|a_2+\frac{1}{c}\right| \leqq 2V\left(\frac{1+|a_2^2-a_3|}{4}\right)$$

As N varies from 0 to $\frac{1}{2}$, the fuuction $V(N)$ increases monotonously from 0 to 1, and the quantity $|a_2^2-a_3| = |b_1|$ never exceeds the unity. Therefore, the right hand side of this inequality is surely significant at any time. Thus, as above, we have first

$$(3.18) \qquad |c| \geqq \frac{1}{|a_2|+2V\left(\frac{1+|a_2^2-a_3|}{4}\right)}$$

for any case, while on the other hand we have

$$(3.19) \qquad |c| \leqq \frac{1}{|a_2|-2V\left(\frac{1+|a_2^2-a_3|}{4}\right)},$$

provided that the denominator on the right hand side is positive; if $|a_2|>2$, it is actually the case, because we have then

$$|a_2|-2V\left(\frac{1+|a_2^2-a_3|}{4}\right) \geqq |a_2|-2>0.$$

Hence, if

$$|a_2|-2V\left(\frac{1+|a_2^2-a_3|}{4}\right) \geqq C>0,$$

then we have from (3.19)

$$(3.20) \qquad |c| \leqq \frac{1}{C},$$

and otherwise we have from (3.18)

$$(3.21) \qquad |c| > \frac{1}{C+4V\left(\frac{1+|a_2^2-a_3|}{4}\right)}.$$

Now, by taking particularly $C=\sqrt{5}-2$, we are led to the Montel-Bieberbach's theorem, because we have then, for the bounds in (3.20) and (3.21), the quantities

$$\frac{1}{C}=\sqrt{5}+2 \quad \text{and} \quad \frac{1}{C+4\sqrt{\left(\frac{1+|a_2^2-a_3|}{4}\right)}}\left(\geqq\frac{1}{C+4}=\sqrt{5}-2\right)$$

respectively. The preciseness of the limits is also a consequence of that of the Landau's limit.