4. On the Flat Conformal Differential Geometry, IV.

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§ 4. *Theory of subspaces.*

We have, in Chapters 1 and 2, established the fundamental differential equations of the flat conformal geometry, and, in Chapter 3, discussed the curves in the flat conformal space and established the Frenet formulae for curves with respect to a projective parameter and with respect to a conformal parameter. In the present Chapter, we shall deal with subspaces in the flat conformal space.

1°. *Subspaces in the flat conformal space*

Let us consider an m-dimensional subspace C_m:

$$\xi^\lambda = \xi^\lambda(\xi^{\dot{1}}, \xi^{\dot{2}}, \ldots\ldots, \xi^{\dot{m}}) \tag{4.1}$$

in the n-dimensional flat conformal space C_n described by a curvilinear coordinates system (ξ^λ). Then, the current point-hypersphere $A_{\dot{0}} = A_0$ on the subspace may be considered as function of m parameters ξ^i $(i, j, k, \ldots\ldots = \dot{1}, \dot{2}, \ldots\ldots, \dot{m})$. Differentiating the relation $A_{\dot{0}} A_{\dot{0}} = 0$, we know that, the hyperspheres

$$A_i = \frac{\partial A_{\dot{0}}}{\partial \xi^i} = \frac{\partial \xi^\lambda}{\partial \xi^i}\frac{\partial A_0}{\partial \xi^\lambda},$$

or

$$A_i = B_i^{\cdot\lambda} A_\lambda \qquad \left(B_i^{\cdot\lambda} = \frac{\partial \xi^\lambda}{\partial \xi^i}\right) \tag{4.2}$$

pass through the point $A_{\dot{0}}$. Moreover, since $dA_{\dot{0}} = d\xi^i A_i$ along the subspace, and consequently each hypersphere A_i belongs to a pencil of hyperspheres determined by the point $A_{\dot{0}}$ and a nearby point $A_{\dot{0}} + dA_{\dot{0}}$ on the subspace, we see that A_i are m hyperspheres orthogonal to the subspace. From (4.2), we have

$$A_j A_k = g_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} g_{\mu\nu}. \tag{4.3}$$

Now, we shall choose $n - m$ mutually orthogonal unit hyperspheres A_P $(P, Q, R, \ldots = \dot{m} + \dot{1}, \ldots, \dot{n})$ all passing through the point $A_{\dot{0}}$ and tangent to the subspace C_m.

Then the hyperspheres A_P, all passing through the point $A_{\dot{0}}$, may be expressed, with respect to the repere $[A_0, A_\lambda, A_\infty]$, in the form

1) K. Yano: On the flat conformal differential geometry, I, II, III. Proc. *21* (1945), 419–429; 454–465; *22* (1946), 9–19.

(4.4) $$A_P = B_P^{\cdot 0} A_0 + B_P^{\cdot\lambda} A_\lambda,$$
where the coefficients satisfy the relations

(4.5) $$g_{\mu\nu} B_j^{\cdot\mu} B_P^{\cdot\nu} = 0, \qquad g_{\mu\nu} B_P^{\cdot\mu} B_Q^{\cdot\nu} = \delta_{PQ}.$$

Finally, we shall denote by $A_{\dot\infty}$ the point of intersection other than $A_{\dot 0}$ of n hyperspheres A_i and A_P such that

$$A_{\dot 0} A_{\dot\infty} = -1.$$

Then the point-hypersphere $A_{\dot\infty}$ will have the expression

(4.6) $$A_{\dot\infty} = \frac{1}{2} B_P^{\cdot 0} B_P^{\cdot 0} A_0 + B_P^{\cdot 0} B_P^{\cdot\lambda} A_\lambda + A_\infty,$$
as it may be easily verified.

The equations $A_{\dot 0} = A_0$, (4.2), (4.4) and (4.6) may be solved with respect to A_0, A_λ and A_∞ as follows:

(4.7) $$\begin{cases} A_0 = A_{\dot 0}, \\ A_\lambda = -B_P^{\cdot 0} B_{P\lambda} A_{\dot 0} + B_{\lambda\cdot}^{i} A_i + B_{P\lambda} A_P, \\ A_\infty = \frac{1}{2} B_P^{\cdot 0} B_P^{\cdot 0} A_{\dot 0} \qquad - B_P^{\cdot 0} A_P + A_{\dot\infty}, \end{cases}$$

where

$$B_{P\lambda} = g_{\lambda\mu} B_P^{\cdot\mu} \quad \text{and} \quad B_{\cdot\lambda}^{i} = g^{ij} g_{\lambda\mu} B_j^{\cdot\mu}.$$

2°. *Fundamental differential equations for subspaces.*

Let us differentiate $n+2$ hyperspheres $A_{\dot 0}$, A_j, A_P and $A_{\dot\infty}$ with respect to the parameters ξ^k, obtaining

$$\frac{\partial A_{\dot 0}}{\partial \xi^k}, \quad \frac{\partial A_j}{\partial \xi^k}, \quad \frac{\partial A_P}{\partial \xi^k}, \quad \frac{\partial A_{\dot\infty}}{\partial \xi^k}.$$

The hyperspheres $A_{\dot 0}$, A_j, A_P and $A_{\dot\infty}$ being linearly independent, they may be considered as forming a repère mobile for the subspace. Consequently, the above hyperspheres must be expressed as linear combinations of the hyperspheres $A_{\dot 0}$, A_j, A_P and $A_{\dot\infty}$ themselves.

First we have

(4.8) $$\frac{\partial A_0}{\partial \xi^k} = A_k.$$

For the hyperspheres $\frac{\partial A_j}{\partial \xi^k}$, we have the relations of the form

(4.9) $$\frac{\partial A_j}{\partial \xi^k} = \Pi_{jk}^{\dot 0} A_{\dot 0} + \Pi_{jk}^{i} A_i + \Pi_{jkQ} A_Q + \Pi_{jk}^{\infty} A_{\dot\infty}.$$

On the other hand, differentiating (4.2) with respect to ξ^k we find

$$\frac{\partial A_j}{\partial \xi^k} = B_{j,k}^{\cdot\lambda} A_\lambda + B_j^{\cdot\mu} B_k^{\cdot\nu} \frac{\partial A_\mu}{\partial \xi^\nu}$$
$$= B_{j,k}^{\cdot\lambda} A_\lambda + B_j^{\cdot\mu} B_k^{\cdot\nu} (\Pi_{\mu\nu}^{0} A_0 + \{^{\lambda}_{\mu\nu}\} A_\lambda + g_{\mu\nu} A_\infty)$$

by virtue of the fundamental equations for C_n, where the comma denotes ordinary differentiation.

Substituting (4.7) in the above equations, we obtain

$$\frac{\partial A_j}{\partial \xi^k} = [B_j^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} - B_P^{\cdot 0} B_{P\lambda} (B_{j,k}^{\cdot\lambda} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) + \frac{1}{2} g_{jk} B_P^{\cdot 0} B_P^{\cdot 0}] A_{\dot 0} + [B^{\cdot i}_{\cdot\lambda} (B_{j,k}^{\cdot\lambda} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\})] A_i + [B_{Q\lambda} (B_{j,k}^{\cdot\lambda} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) - g_{jk} B_Q^{\cdot 0}] A_Q + g_{jk} A_{\dot\infty}.$$

Comparing the coefficients of the corresponding terms in (4.9) and in the above equations, we find

$$(4.10)\quad \begin{cases} \Pi^{\dot 0}_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} - B_P^{\cdot 0} B_{P\lambda} (B_{j,k}^{\cdot\lambda} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) + \frac{1}{2} g_{jk} B_P^{\cdot 0} B_P^{\cdot 0}, \\ \Pi^i_{jk} = B^{\cdot i}_{\cdot\lambda} (B_{j,k}^{\cdot\lambda} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}), \\ \Pi_{jkQ} = B_{Q\lambda} (B_{j,k}^{\cdot\lambda} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) - g_{jk} B_Q^{\cdot 0}, \\ \Pi^{\dot\infty}_{jk} = g_{jk}. \end{cases}$$

We see that the Π^i_{jk} coïncide with the Christoffel symbols $\{^i_{jk}\}$ formed with the components of the fundamental tensor g_{jk} of the subspace C_m.

For the hyperspheres $\frac{\partial A_P}{\partial \xi^k}$, we have the relations of the form

$$(4.11)\quad \frac{\partial A_P}{\partial \xi^k} = \Pi^{\dot 0}_{Pk} A_{\dot 0} + \Pi^{\cdot i}_{\cdot kP} A_i + \Pi_{PQk} A_Q$$

because of the relations $A_{\dot 0} A_P = 0$ and consequently $A_{\dot 0} \frac{\partial A_P}{\partial \xi^k} = 0$.

On the other hand, differentiating (4.4) with respect to ξ^k, we find

$$\frac{\partial A_P}{\partial \xi^k} = B_{P,k}^{\cdot 0} A_0 + B_P^{\cdot 0} A_k + B_{P,k}^{\cdot\lambda} A_\lambda + B_P^{\cdot\mu} B_k^{\cdot\nu} (\Pi^0_{\mu\nu} A_0 + \{^{\lambda}_{\mu\nu}\} A_\lambda)$$

by virtue of the fundamental differential equations for C_n and (4.5).

Substituting (4.7) in the above equations, we obtain

$$\frac{\partial A_P}{\partial \xi^k} = [B_P^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} + B_{P,k}^{\cdot 0} - B_Q^{\cdot 0} B_{Q\lambda} (B_{P,k}^{\cdot\lambda} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\})] A_{\dot 0} + [B^{\cdot i}_{\cdot\lambda} (B_{P,k}^{\cdot\lambda} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) + B_P^{\cdot 0} \delta^i_k] A_i + [B_{Q\lambda} (B_{P,k}^{\cdot\lambda} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\})] A_Q.$$

Comparing the coefficients of the corresponding terms in (4.11) and the above equations, we find

$$(4.12)\quad \begin{cases} \Pi^{\dot 0}_{Pk} = B_P^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} + B_{P,k}^{\cdot 0} - B_Q^{\cdot 0} B_{Q\lambda} (B_{P,k}^{\cdot\lambda} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}), \\ \Pi^{\cdot i}_{\cdot kP} = B^{\cdot i}_{\cdot\lambda} (B_{P,k}^{\cdot\lambda} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) + B_P^{\cdot 0} \delta^i_k, \\ \Pi_{PQk} = (B_{P,k}^{\cdot\lambda} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{^{\lambda}_{\mu\nu}\}) B_{Q\lambda}. \end{cases}$$

Finally, for the hyperspheres $\frac{\partial A_{\dot\infty}}{\partial \xi^k}$, we have the relations of the form

$$(4.13)\quad \frac{\partial A_{\dot\infty}}{\partial \xi^k} = \Pi^{\cdot i}_{\dot\infty k} A_i + \Pi_{\dot\infty Qk} A_Q$$

because of the relations $A_{\dot 0} A_{\dot\infty} = -1$ and $A_{\dot\infty} A_{\dot\infty} = 0$.

From the relations $A_j A_{\dot\infty} = 0$, (4.9) and (4.13), we have

$$(4.14)\quad \Pi^{\;i}_{\dot\infty k} = g^{ij} \Pi^0_{jk}$$

and, from $A_P A_{\dot\infty} = 0$, (4.11), (4.13) and (4.15), (4.15) $$\Pi_{\dot\infty Pk} = \Pi_{Pk}.$$

Thus, we have established the fundamental differential equations for the subspace:

(4.16) $$\begin{cases} \dfrac{\partial A_{\dot 0}}{\partial \xi^k} = A_k, \\ \dfrac{\partial A_j}{\partial \xi_k} = \Pi^{\dot 0}_{jk} A_{\dot 0} + \Pi^i_{jk} A_i + \Pi_{jkQ} A_Q + \Pi^{\dot\infty}_{jk} A_{\dot\infty}, \\ \dfrac{\partial A_P}{\partial \xi^k} = \Pi^{\dot 0}_{Pk} A_{\dot 0} + \Pi^i_{\cdot kP} A_i + \Pi_{PQk} A_Q, \\ \dfrac{\partial A_{\dot\infty}}{\partial \xi^k} = \Pi_{\dot\infty k} A_i + \Pi_{\dot\infty Qk} A_Q, \end{cases}$$

where

(4.17) $$\begin{cases} \Pi^{\dot 0}_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} - B_P^{\cdot 0} H_{jkP} + \dfrac{1}{2} g_{jk} B_P^{\cdot 0} B_P^{\cdot 0}, \quad \Pi^i_{jk} = \{{}^{\,i}_{jk}\}, \\ \Pi_{jkQ} = H_{jkQ} - g_{jk} B_Q^{\cdot 0}, \quad \Pi^{\dot\infty}_{jk} = g_{jk}, \\ \Pi^{\dot 0}_{Pk} = B_P^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} + B^{\cdot 0}_{P,\,k} - B_Q^{\cdot 0} L_{PQk}, \\ \Pi^i_{\cdot kP} = -H^i_{\cdot kP} + \delta^i_k B_P^{\cdot 0}, \quad \Pi_{PQk} = L_{PQk}, \Pi^i_{\dot\infty k} = g^{ij} \Pi^0_{jk}, \Pi_{\dot\infty Qk} = \Pi^0_{Qk}, \end{cases}$$

and

(4.18) $$\begin{cases} H_{jkP} = H_{jk}^{\cdot\cdot\lambda} B_{P\lambda}, \\ H_{jk}^{\cdot\cdot\lambda} = B^{\cdot\lambda}_{j,\,k} + B_j^{\cdot\mu} B_k^{\cdot\nu} \{{}^{\,\lambda}_{\mu\nu}\} - B_i^{\cdot\lambda} \{{}^{\,i}_{jk}\}, \\ L_{PQk} = B^{\cdot\lambda}_{P;\,k} B_{Q\lambda} = (B^{\cdot\lambda}_{P,\,k} + B_P^{\cdot\mu} B_k^{\cdot\nu} \{{}^{\,\lambda}_{\mu\nu}\}) B_{Q\lambda}. \end{cases}$$

Up to the present, the quantities $B_P^{\cdot 0}$ were left undetermined, we shall determine these quantities by invariant condition

$$g^{jk} \frac{\partial A_{\dot 0}}{\partial \xi^j} \frac{\partial A_P}{\partial \xi^k} = 0,$$

which gives

(4.19) $$g^{jk} \Pi_{jkP} = 0.$$

From the expressions for Π_{jkP}, we find

(4.20) $$B_P^{\cdot 0} = \frac{1}{m} g^{jk} H_{jkP} = H_P.$$

Thus, the fundamental differential equations (4.16) take the form

(4.21) $$\begin{cases} \dfrac{\partial A_{\dot 0}}{\partial \xi^k} = A_k, \\ \dfrac{\partial A_j}{\partial \xi^k} = \Pi^{\dot 0}_{jk} A_{\dot 0} + \{{}^{\,i}_{jk}\} A_i + M_{jkQ} A_Q + g_{jk} A_{\dot\infty}, \\ \dfrac{\partial A_P}{\partial \xi^k} = \Pi^0_{Pk} A_{\dot 0} - M^i_{\cdot kP} A_i + L_{PQk} A_Q, \\ \dfrac{\partial A_{\dot\infty}}{\partial \xi^k} = \Pi^{\,i}_{\dot\infty k} A_i + \Pi_{\dot\infty Qk} A_Q, \end{cases}$$

where

$$(4.22)\quad \begin{cases} \overset{0}{\Pi}_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} - H_{jkP} H_P + \frac{1}{2} g_{jk} H_P H_P, \\ M_{jkQ} = H_{jkQ} - g_{jk} H_Q, \\ \overset{0}{\Pi}_{Pk} = B_P^{\cdot\mu} B_k^{\cdot\nu} \Pi^0_{\mu\nu} + H_{P,k} - L_{PQk} H_Q, \\ L_{PQk} = B_{P;k}^{\cdot\lambda} B_{Q\lambda}, \quad \Pi^{\cdot i}_{\infty k} = g^{ij} \overset{0}{\Pi}_{jk}, \quad \Pi^{\cdot}_{\infty Qk} = \overset{0}{\Pi}_{Qk}, \end{cases}$$

and

$$(4.23)\quad M_{jkQ} = M_{jk}^{\cdot\cdot\lambda} B_{Q\lambda} = (H_{jk}^{\cdot\cdot\lambda} - \frac{1}{m} g_{jk} g^{bc} H_{bc}^{\cdot\cdot\lambda}) B_{Q\lambda}.$$

We shall call g_{jk}, M_{jkP} and L_{PQk}, the first, the second and the third fundamental tensors for the subspace respectively, the last two being considered with respect to the tangent hyperspheres A_P here chosen.

It may be observed that the second formulae of (4.21) correspond to the equations of Gauss and the third of (4.21) to the equations of Weingarten in the ordinary differential geometry.

3°. *Remarks on the formation of* A_∞ and A_∞.

In Chapter 1, we have established the fundamental differential equations

$$(4.24)\quad \begin{cases} \frac{\partial A_0}{\partial \xi^\nu} = A_\nu, \\ \frac{\partial A_\mu}{\partial \xi^\nu} = \Pi^0_{\mu\nu} A_0 + \{^{\lambda}_{\mu\nu}\} A_\lambda + g_{\mu\nu} A_\infty, \\ \frac{\partial A_\infty}{\partial \xi^\nu} = \Pi^\lambda_{\infty\nu} A_\lambda \end{cases}$$

for the flat conformal differential geometry, where

$$\Pi^0_{\mu\nu} = -\frac{R_{\mu\nu}}{n-2} + \frac{g_{\mu\nu} R}{2(n-1)(n-2)}, \quad \Pi^\lambda_{\infty\nu} = g^{\lambda\mu} \Pi^0_{\mu\nu}.$$

The A_0 being the current point of the space, the n hyperspheres A_λ passing through the point A_0 are defined by the first of above equations, and the A_∞ is defined as the point of intersection other than A_0 of the n hyperspheres A_λ.

But, from the second of the fundamental differential equations, we see that, the A_0 and A_λ being known, the point-hypersphere A_∞ may also be given by the formula

$$(4.25)\quad A_\infty = \frac{1}{n} g^{\mu\nu} \left(\frac{\partial A_\mu}{\partial \xi^\nu} - \Pi^0_{\mu\nu} A_0 - \{^{\lambda}_{\mu\nu}\} A_\lambda \right).$$

This process of forming A_∞ corresponds to the process of conformal derivative found by T. Y. Thomas.(1)

Starting from the current point A_0, we define $A_\lambda = \frac{\partial A_0}{\partial \xi^\lambda}$, which are n hyperspheres passing through the point A_0, and put $A_\mu A_\nu = g_{\mu\nu}$, then the

(1) T. Y. Thomas: Conformal tensors. (First note). Proc. Nat. Acad. Sci. U.S.A., *18* (1932), 103–112; Conformal tensors. (Second note), ibid. 189–193.

quantities $\Pi^0_{\mu\nu}$ and $\{^{\lambda}_{\mu\nu}\}$ are calculated in terms of $g_{\mu\nu}$, and the hypersphere $\bar{A}_\infty$ defined by

$$\bar{A}_\infty = \frac{1}{n} g^{\mu\nu}\left(\frac{\partial A_\mu}{\partial \xi^\nu} - \Pi^0_{\mu\nu} A_0 - \{^{\lambda}_{\mu\nu}\} A_\lambda\right)$$

is in fact a point-hypersphere and coincide with the A_∞. This fact may be proved as follows: The point A_0 and n hyperspheres A_λ passing through A_0 being thus defined, we denote by A_∞ the point of intersection other than A_0 of n hyperspheres A_λ, then we shall have the fundamental differential equations (4.24) with respect to the repère $[A_0, A_\lambda, A_\infty]$, and consequently (4.25). Then we see that A_∞ and $\bar{A}_\infty$ represent the same point-hypersphere. The same thing may be said for the point-hypersphere $A_{\dot\infty}$.

In the preceding Paragraph of the present Chapter, we have established the fundamental differential equations (4.21) for an m-dimensional subspace in the n-dimensional flat conformal space.

The $A_{\dot 0}$ being the current point on the subspace, the m hyperspheres A_i passing through the point $A_{\dot 0}$ and orthogonal to the subspace are defined by the first of the formulae (4.21), $n-m$ unit hyperspheres A_P passing through the point $A_{\dot 0}$ and orthogonal mutually and to A_i are taken in such a way that we have $g^{ij} A_i \frac{\partial A_P}{\partial \xi^j} = 0$ and finally $A_{\dot\infty}$ is defined as the point of intersection other than $A_{\dot 0}$ of n hyperspheres A_i and A_P.

But, from the second of the fundamental differential equations (4.21) we see that, the $A_{\dot 0}$ and A_i being known, the point-hypersphere $A_{\dot\infty}$ may also be defined by the formula

$$A_{\dot\infty} = \frac{1}{m} g^{jk}\left(\frac{\partial A_j}{\partial \xi^k} - \Pi^{\dot 0}_{jk} A_{\dot 0} - \{^{i}_{jk}\} A_i\right). \tag{4.26}$$

This is a generalization of the process of the conformal derivative of T. Y. Thomas. This method of forming $A_{\dot\infty}$ was used by S. Sasaki(1) in his theory of conformal subspace.

Starting from the current point $A_{\dot 0}$ on the subspace, we define $A_i = \frac{\partial A_{\dot 0}}{\partial \xi^i} = B_i^{\cdot\lambda} A_\lambda$, which are m hyperspheres passing through $A_{\dot 0}$ and orthogonal to the subspace, and put $A_j A_k = g_{jk}$, and next we choose $n-m$ unit hyperspheres $\bar{A}_P = \bar{B}_P^{\cdot 0} A_0 + \bar{B}_P^{\cdot\lambda} A_\lambda$ passing through $A_{\dot 0}$ and orthogonal mutually and to A_i in such a way that we have $g^{ij} A_i \frac{\partial \bar{A}_P}{\partial \xi^j} = 0$, then the quantities $\bar{\Pi}^{\dot 0}_{jk}$, $\{^{i}_{jk}\}$, $\bar{M}_{jkP}$ will be calculated by the formulae (4.22), and the hypersphere $\bar{A}_{\dot\infty}$ defined by

(1) S. Sasaki: Geometry of the conformal connexion. - Science Report of the Tôhoku Imp. Univ. Series I, Vol. XXIX, No. 2, (1940), 219-267.

$$\bar{A}_{\dot{\infty}} = \frac{1}{m} g^{jk}\left(\frac{\partial A_j}{\partial \xi^k} - \bar{\Pi}^{\dot{0}}_{jk} A_{\dot{0}} - \{{}^{\,i}_{jk}\} A_i\right)$$

is in fact a point-hypersphere and coïncides with the $A_{\dot{\infty}}$.

This fact may be proved as follows: The point $A_{\dot{0}}$ and hyperspheres A_i, $\bar{A}_P$ being thus defined, we denote by $\bar{A}_{\dot{\infty}}$ the point of intersection other than $A_{\dot{0}}$ of n hyperspheres A_i and A_P, then we shall have the fundamental differential equations

$$(4.27)\qquad \begin{cases} \dfrac{\partial A_{\dot{0}}}{\partial \xi^k} = A_k, \\ \dfrac{\partial A_j}{\partial \xi^k} = \bar{\Pi}^{\dot{0}}_{jk} A_{\dot{0}} + \{{}^{\,i}_{jk}\} A_i + \bar{M}_{jkQ}\bar{A}_Q + g_{jk} A_{\dot{\infty}}, \\ \dfrac{\partial \bar{A}_P}{\partial \xi^k} = \bar{\Pi}^{\dot{0}}_{Pk} A_{\dot{0}} - \bar{M}^{\cdot i}_{\cdot kP} A_i + \bar{L}_{PQk} A_Q, \\ \dfrac{\partial \bar{A}_{\dot{\infty}}}{\partial \xi^k} = \bar{\Pi}^{i}_{\dot{\infty}k} A_i + \bar{\Pi}_{\dot{\infty}Qk} \bar{A}_Q \end{cases}$$

with respect to the repère $[A_{\dot{0}}, A_i, \bar{A}_P, \bar{A}_\infty]$, where

$$\begin{cases} \bar{\Pi}^{\dot{0}}_{jk} = B^{\cdot\mu}_j B^{\cdot\nu}_k \Pi^{0}_{\mu\nu} - \bar{H}_{jkP}\bar{H}_P + \dfrac{1}{2} g_{jk} \bar{H}_P \bar{H}_P, \\ \bar{M}_{jkQ} = \bar{H}_{jkQ} - g_{jk}\bar{H}_Q. \end{cases}$$

But we have

$$A_P = a^{\cdot 0}_P A_{\dot{0}} + a_{PQ} A_Q$$

where

$$a_{PQ}\, a_{RQ} = \delta_{PR}.$$

Consequently, we have

$$\bar{B}^{\cdot 0}_P = a^{\cdot 0}_P + a_{PQ} B^{\cdot\lambda}_Q, \quad \bar{B}^{\cdot\lambda}_P = a_{PQ} B^{\cdot\lambda}_Q,$$

from which

$$\bar{H}_{jkP}\, a_{PQ} = H_{jkQ}, \quad \bar{H}_P\, a_{PQ} = H_Q, \quad \bar{M}_{jkP}\, a_{PQ} = M_{jkQ}.$$

Thus we see that

$$\bar{H}_{jkP}\bar{H}_P = H_{jkP} H_P \quad \text{and} \quad \bar{H}_P \bar{H}_P = H_P H_P,$$

and the quantities $\bar{\Pi}^{\dot{0}}_{jk}$ and $\Pi^{\dot{0}}_{jk}$ coïncide.

Consequently, we have, from the second of the fundamental differential equations (4.27),

$$\bar{A}_{\dot{\infty}} = \frac{1}{m} g^{jk}\left(\frac{\partial A_j}{\partial \xi^k} - \Pi^{\dot{0}}_{jk} A_{\dot{0}} - \{{}^{\,i}_{jk}\} A_i\right)$$

by virtue of the relations $g^{jk} M_{jkQ} = 0$, and we can conclude that $A_{\dot{\infty}}$ and $\bar{A}_{\dot{\infty}}$ coïncide.

4°. *Integrability conditions of the fundamental differential equations for subspaces.*

In this section, we shall investigate the integrability conditions of the fundamental differential equations (4.21) for the subspaces.

We shall first observe that the coefficients of the second of the equations (4.21) must be symmetric with respect to j and k, thus

$$\text{(4.28)} \qquad \Pi^{\dot 0}_{jk} = \Pi^{\dot 0}_{kj}, \quad \left\{{i \atop jk}\right\} = \left\{{i \atop kj}\right\}, \quad M_{jkQ} = M_{kjQ}, \quad g_{jk} = g_{kj},$$

which are naturally satisfied by virtue of their definitions (4.22).

Calculating the $\dfrac{\partial^2 A_j}{\partial\xi^k \partial\xi^h} - \dfrac{\partial^2 A_j}{\partial\xi^h \partial\xi^k} = 0$ and substituting the relations (4.21) themselves in the resulting relations, we find

$$\text{(4.29)} \qquad \Pi^{\dot 0}_{jk,h} - \Pi^{\dot 0}_{jh,k} + \left\{{a \atop jk}\right\} \Pi^{\dot 0}_{ah} - \left\{{a \atop jh}\right\} \Pi^{\dot 0}_{ak} + M_{jkP}\Pi^{\dot 0}_{Ph} - M_{jhP}\Pi^{\dot 0}_{Pk} = 0,$$

$$\text{(4.30)} \qquad \left\{{i \atop jk}\right\}_{,h} - \left\{{i \atop jh}\right\}_{,k} + \left\{{a \atop jk}\right\}\left\{{i \atop ah}\right\} - \left\{{a \atop jh}\right\}\left\{{i \atop ak}\right\} + \Pi^{\dot 0}_{jk}\delta^i_h - \Pi^{\dot 0}_{jh}\delta^i_k$$
$$+ g_{jk}\Pi^{\ i}_{\dot\infty h} - g_{jh}\Pi^{\ i}_{\dot\infty k} - M_{jkP}M^{\cdot i}_{\cdot hP} + M_{jhP}M^{\cdot i}_{\cdot kP} = 0,$$

$$\text{(4.31)} \qquad M_{jkQ,h} - M_{jhQ,k} + \left\{{a \atop jk}\right\} M_{ahQ} - \left\{{a \atop jh}\right\} M_{akQ}$$
$$+ M_{jkP}L_{PQh} - M_{jhP}L_{PQk} + g_{jk}\Pi_{\dot\infty Qh} - g_{jh}\Pi_{\dot\infty Qk} = 0,$$

$$\text{(4.32)} \qquad g_{jk,h} - g_{jh,k} + \left\{{a \atop jk}\right\} g_{ah} - \left\{{a \atop jh}\right\} g_{ak} = 0,$$

by virtue of the linear independence of $A_{\dot 0}$, A_i, A_P and $A_{\dot\infty}$.

Denoting by a semi-colon the covariant derivative with respect to the Christiffel symbols $\left\{{i \atop jk}\right\}$, we have, from these equations,

$$\text{(4.33)} \qquad C^{\dot 0}_{\cdot jkh} + M_{jkP}\Pi^{\dot 0}_{Ph} - M_{jhP}\Pi^{\dot 0}_{Pk} = 0,$$

$$\text{(4.34)} \qquad C^{i}_{\cdot jkh} - M_{jkP}M^{\cdot i}_{\cdot hP} + M_{jhP}M^{\cdot i}_{\cdot kP} = 0,$$

$$\text{(4.35)} \qquad M_{jkQ;h} - M_{jhQ;k} + M_{jkP}L_{PQh} - M_{jhP}L_{PQk} + g_{jk}\Pi_{\dot\infty Qh} - g_{jh}\Pi_{\dot\infty Qk} = 0,$$

the equations (4.32) being reduced to identities, where we have put

$$\text{(4.36)} \qquad C^{\dot 0}_{\cdot jkh} = \Pi^{\dot 0}_{jk;h} - \Pi^{\dot 0}_{jh;k},$$

$$\text{(4.37)} \qquad C^{i}_{\cdot jkh} = R^{i}_{\cdot jkh} + \Pi^{\dot 0}_{jk}\delta^i_h - \Pi^{\dot 0}_{jh}\delta^i_k + g_{jk}\Pi^{\ i}_{\dot\infty h} - g_{jh}\Pi^{\ i}_{\dot\infty k},$$

and $R^{i}_{\cdot jkh}$ are components of the Riemann-Christoffel curvature tensor formed with g_{jk}.

Calculating next $\dfrac{\partial^2 A_P}{\partial\xi^k \partial\xi^h} - \dfrac{\partial^2 A_P}{\partial\xi^h \partial\xi^k} = 0$ and substituting the relations (4.21) in the resulting relations, we find

$$\text{(4.38)} \qquad \Pi^{\dot 0}_{Pk;h} - \Pi^{\dot 0}_{Ph;k} + M^{\cdot a}_{\cdot kP}\Pi^{\dot 0}_{ah} - M^{\cdot a}_{\cdot hP}\Pi^{\dot 0}_{ak} + L_{PQk}\Pi^{\dot 0}_{Qh} - L_{PQh}\Pi^{\dot 0}_{Qk} = 0,$$

$$\text{(4.39)} \qquad M^{\cdot i}_{\cdot kP;h} - M^{\cdot i}_{\cdot hP;k} + M^{\cdot i}_{\cdot kQ}L_{QPh} - M^{\cdot i}_{\cdot hQ}L_{QPk} + \delta^i_k\Pi^{\dot 0}_{Ph} - \delta^i_h\Pi^{\dot 0}_{Pk} = 0,$$

$$\text{(4.40)} \qquad L_{PQk;h} - L_{PQh;k} + M^{\cdot a}_{\cdot kP}M_{ahQ} - M^{\cdot a}_{\cdot hP}M_{akQ}$$
$$+ L_{PRk}L_{RQh} - L_{PRh}L_{RQk} = 0$$

because of the linear independence of $A_{\dot 0}$, A_i, A_P and $A_{\dot\infty}$. The equations (4.39) coïncide with (4.35).

Calculating finally $\dfrac{\partial^2 A_{\dot\infty}}{\partial\xi^k \partial\xi^h} - \dfrac{\partial^2 A_{\dot\infty}}{\partial\xi^h \partial\xi^k} = 0$ and substituting (4.21) in the resulting relations, we find

$$\text{(4.41)} \qquad \Pi^{\ i}_{\dot\infty k;h} - \Pi^{\ i}_{\dot\infty h;k} - \Pi_{\dot\infty Pk}M^{\cdot i}_{\cdot hP} + \Pi_{\dot\infty Ph}M^{\cdot i}_{\cdot kP} = 0,$$

$$\text{(4.42)} \qquad \Pi_{\dot\infty Qk;h} - \Pi_{\dot\infty Qh;k} - \Pi^{\ a}_{\dot\infty k}M_{ahQ} + \Pi^{\ a}_{\dot\infty h}M_{akQ}$$
$$- \Pi_{\dot\infty Pk}L_{PQh} + \Pi_{\dot\infty Ph}L_{PQk} = 0$$

because of the linear independence of $A_{\dot{0}}$, A_i, A_P and $A_{\dot{\infty}}$. The equations (4.41) coïncide with (4.33) and (4.42) with (4.38).

Thus, as integrability conditions of the fundamental differential equations (4.21), we have obtained

$$(4.43)\quad \begin{cases} C^{\dot{0}}_{\cdot jkh} + M_{jkP}\Pi^{\dot{0}}_{Ph} - M_{jhP}\Pi^{\dot{0}}_{Pk} = 0, \\ C_{\cdot jkh} - M_{jkP}M^{\dot{i}}_{\cdot hP} + M_{jhP}M^{\dot{i}}_{\cdot kP} = 0, \\ M_{jkQ;h} - M_{jhQ;k} + M_{jkP}L_{PQh} - M_{jhP}L_{PQk} + g_{jk}\Pi_{\dot{\infty}Qh} - g_{jh}\Pi_{\dot{\infty}Qk} = 0, \\ \Pi^{\dot{0}}_{Pk;h} - \Pi^{\dot{0}}_{Ph;k} + M^{a}_{\cdot kP}\Pi^{\dot{0}}_{ah} - M^{a}_{\cdot hP}\Pi^{\dot{0}}_{ak} \\ \qquad\qquad + L_{PQk}\Pi^{\dot{0}}_{Qh} - L_{PQh}\Pi^{\dot{0}}_{Qk} = 0, \\ L_{PQk;h} - L_{PQh;k} + M^{a}_{\cdot kP}M_{ahQ} - M^{a}_{\cdot hP}M_{akQ} \\ \qquad\qquad + L_{PRk}L_{RQh} - L_{PRh}L_{PQk} = 0. \end{cases}$$

The second, third and fifth of these equations correspond respectively to the equations of Gauss, Codazzi and Ricci in the ordinary differential geometry.

5°. *The fundamental theorèm of subspace theory.*

In the preceding Paragraph, we have seen that, the first, second and third fundamental tensors g_{jk}, M_{jkP} and L_{PQk} of an m-dimensional subspace C_m appearing in the fundamental differential equations

$$(4.44)\quad \begin{aligned} \frac{\partial A_{\dot{0}}}{\partial \xi^k} &= A_k, \\ \frac{\partial A_j}{\partial \xi^k} &= \Pi^{\dot{0}}_{jk}A_{\dot{0}} + \{{}_{jk}^{\,i}\}A_i + M_{jkQ}A_P + g_{jk}A_{\dot{\infty}}, \\ \frac{\partial A_P}{\partial \xi^k} &= \Pi^{\dot{0}}_{Pk}A_{\dot{0}} - M^{i}_{\cdot kP}A_i + L_{PQk}A_Q, \\ \frac{\partial A_{\dot{\infty}}}{\partial \xi^k} &= \Pi^{i}_{\dot{\infty}k}A_i + \Pi_{\dot{\infty}Qk}A_Q \end{aligned}$$

must satisfy the equations (4.43).

Putting $i = h = a$ in the second of the equations (4.43) and summing up for a from $\dot{1}$ to $\dot{m}$, we find

$$(4.45)\quad R_{jk} + (m-2)\Pi^{\dot{0}}_{jk} + g_{jk}g^{bc}\Pi^{\dot{0}}_{bc} + M_{jaP}M^{a}_{\cdot kP} = 0$$

by virtue of (4.37) and $M^{a}_{\cdot aP} = 0$, where $R_{jk} = R^{a}_{\cdot jka}$.

Contracting g^{jk} to (4.45), we obtain

$$R + 2(m-1)g^{bc}\Pi^{\dot{0}}_{bc} + M^{a}_{\cdot bP}M^{b}_{\cdot aP} = 0,$$

from which

$$g^{bc}\Pi^{\dot{0}}_{bc} = -\frac{R}{2(m-1)} - \frac{M^{a}_{\cdot bP}M^{b}_{\cdot aP}}{2(m-1)}$$

where $R = g^{jk}R_{jk}$. Substituting this ueqation into (4.45), we find

$$(4.46)\quad \Pi^{\dot{0}}_{jk} = -\frac{R_{jk}}{m-2} + \frac{g_{kj}R}{2(m-1)(m-2)} - \frac{M_{jaP}M^{a}_{\cdot kP}}{m-2} + \frac{g_{jk}(M^{a}_{\cdot bP}M^{b}_{\cdot aP})}{2(m-1)(m-2)}$$

and we see that $\overset{0}{\Pi}{}_{jk}$ are calculated exclusively in terms of the components of the first and second fundamental tensors g_{jk} and M_{jkP}. Contracting next g^{jk} to the third equations of (4.43), we obtain

$$-M^{a}_{\cdot hQ;a}-M^{a}_{\cdot hP}L_{PQa}+(m-1)\Pi_{\dot{\infty}Qh}=0,$$

from which

$$\Pi_{\dot{\infty}Qh}=\overset{0}{\Pi}{}_{Qh}=\frac{1}{m-1}(M^{a}_{\cdot hQ;a}+M^{a}_{\cdot hP}L_{PQa}) \tag{4.47}$$

and we see that $\Pi_{\dot{\infty}Qh}=\overset{0}{\Pi}{}_{Qh}$ are expressed in terms of the components of the first, second and third fundamental tensors.

Substituting (4.46) and (4.47) in (4.43), we obtain

$$\left\{\begin{aligned}
&\left[\frac{R_{jk;h}}{m-2}-\frac{g_{jk}R_{;h}}{2(m-1)(m-2)}+\frac{(M_{jaP}M^{a}_{\cdot kP})_{;h}}{m-2}-\frac{g_{jk}(M^{a}_{\cdot bP}M^{b}_{\cdot aP})_{;h}}{2(m-1)(m-2)}\right]\\
&-\left[\frac{R_{jh;k}}{m-2}-\frac{g_{jh}R_{;k}}{2(m-1)(m-2)}+\frac{(M_{jaP}M^{a}_{\cdot hP})_{;k}}{m-2}-\frac{g_{jh}(M^{a}_{\cdot bP}M^{b}_{\cdot aP})_{;k}}{2(m-1)(m-2)}\right]\\
&-\frac{1}{m-1}M_{jkP}(M^{a}_{\cdot hP;a}+M^{a}_{\cdot hQ}L_{QPa})+\frac{1}{m-1}M_{jhP}(M^{a}_{\cdot kP;a}\\
&+M^{a}_{\cdot kQ}L_{QPa})=0,\\
&R^{\cdot}_{\cdot jkh}-\frac{1}{m-2}(R_{jk}\delta^{i}_{h}-R_{jh}\delta^{i}_{k}+g_{jk}R^{i}_{\cdot h}-g_{jh}R^{i}_{\cdot k})\\
&+\frac{R}{(m-1)(m-2)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})=M_{jkP}M^{i}_{\cdot hP}-M_{jhP}M^{i}_{\cdot kP}\\
&+\frac{1}{m-2}\Big[M_{jaP}M^{a}_{\cdot kP}\delta^{i}_{h}-M_{jaP}M^{a}_{\cdot hP}\delta^{i}_{k}+g_{jk}M^{i}_{\cdot aP}M^{a}_{\cdot hP}\\
&-g_{jh}M^{i}_{\cdot aP}M^{a}_{\cdot kP}\Big]-\frac{M^{a}_{\cdot bP}M^{b}_{\cdot aP}}{(m-1)(m-2)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k}),\\
&M_{jkQ;h}-M_{jhQ;k}+M_{jkP}L_{PQh}-M_{jhP}L_{PQk}\\
&+\frac{1}{m-1}g_{jk}(M^{a}_{\cdot hQ;a}+M^{a}_{\cdot hR}L_{RQa})\\
&-\frac{1}{m-1}g_{jh}(M^{a}_{\cdot kQ;a}+M^{a}_{\cdot kP}L_{RQa})=0,\\
&\frac{1}{m-1}(M^{a}_{\cdot kP;a}+M^{a}_{\cdot kQ}L_{QPa})_{;h}-\frac{1}{m-1}(M^{a}_{\cdot hP;a}+M^{a}_{\cdot hQ}L_{QPa})_{;k}\\
&+\frac{1}{m-1}(M^{a}_{\cdot kQ;a}+M^{a}_{\cdot kR}L_{RQa})L_{QPh}-\frac{1}{m-1}(M^{a}_{\cdot hQ;a}\\
&+M^{a}_{\cdot hR}L_{RQa})L_{QPk}-\frac{1}{m-2}(R_{ik}M^{i}_{\cdot hP}-R_{ih}M^{i}_{\cdot kP}\\
&+M_{iaQ}M^{a}_{\cdot kQ}M^{i}_{\cdot hQ}-M_{iaQ}M^{a}_{\cdot hQ}M^{i}_{\cdot kQ})=0,\\
&L_{PQk;h}-L_{PQh;k}+M^{a}_{\cdot kP}M_{ahQ}-M^{a}_{\cdot hP}M_{akQ}\\
&+L_{PRk}L_{RQh}-L_{PRh}L_{RQk}=0.
\end{aligned}\right. \tag{4.48}$$

These are the necessary and sufficient conditions that the fundamental differential equations of the subspace are completely integrable, $\Pi^{0}{}_{k}$ and $\overset{0}{\Pi}{}_{Pk}=\Pi_{\dot{\infty}Pk}$ being given by (4.46) and (4.47) respectively.

Now, suppose that the three tensors g_{jk}, M_{jkP} and L_{PQk} satisfy the equations (4.48), then the differential equations of the form (4.44) are completely integrable, the $\Pi^{\dot{0}}_{jk}$ and $\Pi^{0}_{Pk} = \Pi_{\dot{\infty}Pk}$ being respectively given by (4.46) and (4.47).

Thus if we give a system of initial values $(A_{\dot{0}})_0$, $(A_i)_0$, $(A_P)_0$, $(A_{\dot{\infty}})_0$ of $A_{\dot{0}}$, A_i, A_P, $A_{\dot{\infty}}$ at a fixed point of the space, then the $A_{\dot{0}}$, A_i, A_P, $A_{\dot{\infty}}$ satisfying (4.44) are completely determined. But if we put

$$T_{\dot{0}\dot{0}} = A_{\dot{0}} A_{\dot{0}}, \quad T_{\dot{0}j} = A_{\dot{0}} A_j, \quad T_{\dot{0}P} = A_{\dot{0}} A_P, \quad T_{\dot{0}\dot{\infty}} = A_{\dot{0}} A_{\dot{\infty}} + 1,$$
$$T_{ij} = A_i A_j - g_{ij}, \quad T_{iP} = A_i A_P, \quad T_{i\dot{\infty}} = A_i A_{\dot{\infty}},$$
$$T_{PQ} = A_P A_Q - \delta_{PQ}, \quad T_{P\dot{\infty}} = A_P A_{\dot{\infty}}, \quad T_{\dot{\infty}} T_{\dot{\infty}} = A_{\dot{\infty}} A_{\dot{\infty}},$$

we have

$$\frac{\partial T_{\dot{0}\dot{0}}}{\partial \xi^k} = 2\, T_{\dot{0}k},$$
$$\frac{\partial T_{\dot{0}j}}{\partial \xi^k} = T_{jk} + \Pi^{\dot{0}}_{jk} T_{\dot{0}\dot{0}} + \left\{ {i \atop jk} \right\} T_{\dot{0}i} + M_{jkP} T_{\dot{0}P} + g_{jk} T_{\dot{0}\dot{\infty}},$$
$$\frac{\partial T_{\dot{0}P}}{\partial \xi^k} = T_{kP} + \Pi^{0}_{Pk} T_{\dot{0}\dot{0}} - M^{i}_{\cdot kP} T_{\dot{0}i} + L_{PQk} T_{\dot{0}Q},$$
$$\frac{\partial T_{\dot{0}\dot{\infty}}}{\partial \xi^k} = T_{k\dot{\infty}} + \Pi^{i}_{\dot{\infty}k} T_{\dot{0}i} + \Pi_{\dot{\infty}Qk} T_{\dot{0}Q},$$
$$\frac{\partial T_{ij}}{\partial \xi^k} = \Pi^{0}_{ik}\, {}_{\dot{0}j}T + \left\{ {a \atop ik} \right\} T_{aj} + M_{ikP} T_{jP} + g_{ik} T_{j\dot{\infty}}$$
$$+ \Pi^{0}_{jk} T_{\dot{0}i} + \left\{ {a \atop jk} \right\} T_{ai} + M_{jkP} T_{iP} + g_{jk} T_{i\dot{\infty}},$$
$$\frac{\partial T_{iP}}{\partial \xi^k} = \Pi^{\dot{0}}_{ik} T_{\dot{0}P} + \left\{ {a \atop ik} \right\} T_{aP} + M_{ikQ} T_{QP} + g_{ik} T_{P\dot{\infty}}$$
$$+ \Pi^{\dot{0}}_{Pk} T_{\dot{0}i} - M^{a}_{\cdot kP} T_{ai} + L_{PQk} T_{iQ},$$
$$\frac{\partial T_{i\dot{\infty}}}{\partial \xi^k} = \Pi^{\dot{0}}_{ik} T_{\dot{0}\dot{\infty}} + \left\{ {a \atop ik} \right\} T_{a\dot{\infty}} + M_{ikP} T_{P\dot{\infty}} + g_{ik} T_{\dot{\infty}\dot{\infty}}$$
$$+ \Pi^{a}_{\dot{\infty}k} T_{ai} + \Pi_{\dot{\infty}Qk} T_{iQ},$$
$$\frac{\partial T_{PQ}}{\partial \xi^k} = \Pi^{\dot{0}}_{Pk} T_{\dot{0}Q} - M^{i}_{\cdot kP} T_{iQ} + L_{PRk} T_{RQ}$$
$$+ \Pi^{\dot{0}}_{Qk} T_{\dot{0}P} - M^{i}_{\cdot kQ} T_{iP} + L_{QRk} T_{RP},$$
$$\frac{\partial T_{P\dot{\infty}}}{\partial \xi^k} = \Pi^{\dot{0}}_{Pk} T_{\dot{0}\dot{\infty}} - M^{i}_{\cdot hP} T_{i\dot{\infty}} + L_{PQk} T_{Q\dot{\infty}}$$
$$+ \Pi^{i}_{\dot{\infty}k} T_{iP} + \Pi_{\dot{\infty}Qk} T_{QP},$$
$$\frac{\partial T_{\dot{\infty}\dot{\infty}}}{\partial \xi^k} = 2(\Pi^{i}_{\dot{\infty}k} T_{i\dot{\infty}} + \Pi_{\dot{\infty}Qk} T_{Q\dot{\infty}}),$$

which show that the first partial derivatives of T's are linear homogeneous functions of T's. Consequently the second, the third, ... partial derivatives of T's are also linear homogeneous functions of T's. Thus, if we choose a system of solutions of (4.44) whose initial values satisfy $T = 0$, the conditions $T = 0$ will be always satisfied by the solutions, that is to say, if we fix an initial repère $[(A_{\dot{0}})_0, (A_i)_0, (A_P)_0, (A_{\dot{\infty}})_0]$ at a point of the space, the solutions of the differential equations (4.44), whose coefficients satisfy the conditions (4.48), always exist and constitute a moving repère $[A_{\dot{0}}, A_i, A_P, A_{\dot{\infty}}]$ of an

m-dimensional subspace described by $A_{\dot{0}}$, which coincides with $[(A_{\dot{0}})_0, (A_i)_0, (A_P)_0, (A_{\dot{\infty}})_0]$ at the given point. Two such initial reperes given at different points being always superposed each on the other by a certain conformal transformation, we have proved the

Fundamental theorem of subspace theory[1]*:* If we are given three tensors $g_{jk}(=g_{kj})$, $M_{jkP}(=M_{kjP}, g^{jk}M_{jkP}=0)$ and $L_{PQk}(=-L_{QPk})$ satisfying the conditions (4.48), there exists always a subspace whose first, second and third fundamental tensors are respectively g_{jk}, M_{jkP} and L_{PQk}, two such subspaces being always capable to be superposed by a certain conformal transformation.

1) K Yano and Y. Muto: Sur le théorème fondamental dans la géométrie conforme des sous-espaces riemanniens. Proc. Physico-Math. Soc. Japan, *24* (1942), 437–449.