46. On the Unitary Equivalence in Genral Euclid Space.

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I. *Introduction and the theorem.* The problem of the unitary equivalence of two bounded self-adjoint (s. a.) operators in Hilbert space was solved by E. Hellinger(1) and H. Hahn;(2) the result was extended by M. H. Stone(3) to the case of not necessarily bounded s. a. operators. Later, K. Friedrichs(4) and H. Nakano(5) obtained respectively new forms of the condition for the unitary equivalence; and their results were respectively extended by F. Wecken(6) and H. Nakano(7) to the case of general euclid space R (the space in which all the axioms of the Hilbert space are satisfied except the axiom of separability). The purpose of the present note is to give a condition of tne unitary equivalence in a form somewhat more simple and more algebraical than those of the above cited authors. It is easy to see(8) that we may reduce the problem to the case of bounded s. a. operators T_1 and T_2. For any bonnded s. a. operator T let (T)' be he totality of the bounded linear operators commutative with T, and let (T)" be the totality of the bounded linear operators commutative with every operator ε (T)'. Then (T)' and (T)" are operator rings (with complex multipliers) and satisfy the condition (1) if $S \varepsilon (T)'$ ((T)") the conjugate operator S^* also ε (T)' ((T)").

Moreover the ring (T)" is commutative. In terms of the operator-ring theory our result reads as follows.

Theorem. For the unitary equivalence of T_1 and T_2 it is necessary and sufficient that the ring $(T_1)'$ is isomorphic (with complex multipliers) to the ring $(T_2)'$ by a correspondence C which maps T_1 onto T_2 and which maps conjugate operators onto conjugate operators.

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(1) Dissertation, Göttingen' 1907.
(2) Monatsheft Math. u. Phys. *23* (1912), 169–224.
(3) Linear transformations in Hilbert space, New York 1932.
(4) Jahresber. d D. Math. Ver. *45* (1935) II, 79–82.
(5) Ann. of Math. *42* (1941), 657–664.
(6) Math. Ann. *116* (1939), 422–455.
(7) Math. Ann. 118 (1941), 112–133.
(8) Consider $\mathrm{Tan}^{-1} T_1$ and $\mathrm{Tan}^{-1} T_2$ if T_1 and T_2 are unbounded.

2. *Proof of the theorem.* The necessity is evident. We will prove the sufficiency. The isomorphism C maps s. a. operators onto s. a. operators and positive definite operators onto positive definite operators. The latter fact may be proved by taking the square root of the positive definite operator. We will wri e $A \geqq B$ if the operator (A-B) is positive definite. Let $\{T_{1n}\}$ be a sequence of s. a. operators $\epsilon (T_1)''$ such that $T_{11} \leqq T_{12} \leqq \ldots \leqq T_{1n} \leqq \ldots$ a s. a. operators ϵ $(T_1)''$, and let $T_{1n} \longleftrightarrow T_{2n}$ by the isomorphism C, then we have

(2) strong $\lim_{n\to\infty}$ limit T_{1n} = strong $\lim_{n\to\infty}$ limit T_{2n} by C

This results from the fact that the strong limit$_{n\to\infty}$ $T_{1n} = \sup_{n\geqq 1} T_{1n}$ in $(T_1)''$ (in the sense of the semi-order $\geqq$), and hence the strong limit$_{n\to\infty}$ $T_{2n} = \sup_{n\geqq 1} T_{2n}$ in $(T_2)''$. Thus we have the

Lemma. Let $T_1 = \int \lambda dE_1(\lambda)$ and $T_2 = \int \lambda dE_2(\lambda)$ be the spectral resolution of T_1 and T_2, then if $G(\lambda)$ denotes the characteristic function of a Borel measurable set $\mathfrak{A}$ on $(-\infty, \infty)$

(3) $G(T_1) = \int_{\mathfrak{A}} G(\lambda) dE_1(\lambda) \longleftrightarrow G(T_2) = \int_{\mathfrak{A}} G(\lambda) dE_2(\lambda)$ by C.

It is easy to see, by the isomorphism C, that the dimensions of the closed linear manifolds $N(T_1) = \{x;\ T_1 x = 0\}$, $N(T_2) = \{y;\ T_2 y = 0\}$ are the same. We put, for any $x \epsilon R \ominus N(T_1)$

$M_{T_1}(x) = \{F(T_1)x = \int F(\lambda) dE_1(\lambda)x;\ \int |F(\lambda)|^2 d \| E_1(\lambda)x \|^2 < \infty$, where $F(\lambda)$ denote complex-valued Borel measurable functions$\}$

As is well-known, $M_{T_1}(x)$ is a separable closed linear manifold determined by the set of elements $\{E_1(\lambda)x\}$, $-\infty < \lambda < \infty$; it reduces both $E_1(\lambda)$ and T_1 viz. the projection $P(M_{T_1}(x))$ upon the manifold $M_{T_1}(x)$ is commutative with $E_1(\lambda)$ and with T_1. Let $P_2 \epsilon (T_2)'$ be the operator which corresponds to $P_1 = P(M_{T_1}(x))$ by the isomorphism C, then P_2 is also a projection and $P_2 R \leqq R \ominus N(T_2)$. As $M_{T_1}(x')$ is orthogonal to $M_{T_1}(x)$ if x' is in $R \ominus N(T_1)$ and ortho gonal to $M_{T_1}(x)$, our theorem will be proved if we show that there exists an isometric mapping V from P_1R onto P_2R such that

(4) $P_1 T_1 P_1 = V^{-1} P_2 T_2 P_2 V.$

First we will show that the closed linear manifold $M = P_2R$ is separable. —Proof. Lett $\{y_\alpha\}$ be a complete orthonormal system in P_2R, and we classi-

(9) The existence of the strong limit$_{n\to\infty}$ T_{1n} may be proved following F. Riesz's idea. See the footnote in K. Yosida and T. Nakayama: Proc. Imp. Acad Tokyo, 18 (1942), 555-560.

(10) Acta Sci. Math. Szeged, 7 (1935), 147-159.

fy the set $\{M_{T_2}(y_\alpha)\}$ as follows; $M_{T_2}(y_\alpha)$ and $M_{T_2}(y_\beta)$ belong to the same class if and only if there exists a finite number of elements $y_{\alpha_1} = y_\alpha$, $y_{\alpha_2} \ldots$, $y_{\alpha_n} = y_\beta$ such that $M_{T_2}(y_{\alpha_{i+1}})$ is not orthogonal to $M_{T_2}(y_{\alpha_i})$. Let the set of these classes k be K, then the closed linear manifold $M^{(k)}$ spanned by $M_{T_2}(y_\alpha) \in k$ is a separable closed linear manifold $\leqq P_2R$ which reduces T_2 and P_2. Clearly $P_2R = \sum_{k \in K} M^{(k)}$; here the cardinal number of K must be at most $\aleph_0$. This results from the fact that since $M_{T_1}(x)$ is separable there exists at most countable number of mutually orthogonal projections $P(1) \in (T_1)'$ which satisfy $P(1)P_1 = P_1P(1)$ and hence, because of the isomorphism C, there exists at most countable number of mutually orthogonal projections $P(2) \in (T_2)'$ which satisfy $P(2)P_2 = P_2P(2) = P(2)$.

As P_2R is separable, there exists an element $y \in P_2R$ such that, for any $z \in P_2R$, the monotone increasing function $\sigma(\lambda) = \| E_2(\lambda) z \|^2$ is absolutely continuous with respect to the monotone increasing function $\kappa(\lambda) = \| E_2(\lambda) y \|^2$. We will show that $M_{T_2}(y) = P_2R$. Proof. If otherwise, the projection $P(M_{T_2}(y))$ satisfies

(5) $\quad P_2P(M_{T_2}(y)) = P(M_{T_2}(y))P_2 = P(M_{T_2}(y)) \neq P_2.$

Let Q be the projection $\in (T_1)'$ which corresponds to $P(M_{T_2}(y))$ by the isomorphism C, then we have

(6) $\quad 0 \neq Q = QP_1 = P_1Q \neq P_1.$

Since QR is separable, there exists $x^{(1)} \in QR$ such that, for any $z^{(1)} \in QR$, $\sigma^{(1)}(\lambda) = \| E_1(\lambda) z^{(1)} \|^2$ is absolutely continuous with respect to $\kappa^{(1)}(\lambda) = \| E_1(\lambda) x^{(1)} \|^2$. Then there exists Borel measurable set $\mathfrak{A}$ such that

(7) $$\int_{\mathfrak{A}} d \| E_1(\lambda) x \|^2 \neq 0, \quad \int_{\mathfrak{A}} d \| E_1(\lambda) x^{(1)} \|^2 = 0.$$

For, if otherwise, $\rho_1(\lambda) = \| E_1(\lambda) x \|^2$ is absolutely continuous with respect to $\kappa^{(1)}(\lambda)$. And since $\kappa^{(1)}(\lambda)$ is absolutely continuous with respect to $\rho_1(\lambda)$ by $Q = P_1 Q = QP_1$, we would have $M_{T_1}(x) = M_{T_1}(x^{(1)})$ viz. $Q = P_1$, contrary to (6). Let $G(\lambda)$ be the characteristic function of $\mathfrak{A}$ then we have from (7)

$$G(T_1) x \neq 0, \quad G(T_1) x^{(1)} = 0.$$

Hence we have $G(T_1)P_1 \neq 0$ and, for any $z^{(1)} \in QR$, $G(T_1) z^{(1)} = 0$ or $G(T_1)Q = 0$, because $\sigma^{(1)}(\lambda)$ is of the form $\int_{-\infty}^{\lambda} F(\lambda) d\, \kappa^{(1)}(\lambda)$ and thus $\| G(T_1) z^{(1)} \|^2 = \int F(\lambda) d \| E_1(\lambda) x^{(1)} \|^2 = 0$. Therefore, by (3), $G(T_2)P_2 \neq 0$ and $G(T_2)P(M_{T_2}(y)) = 0$. This contradicts to the choice of y. Hence we must have $M_{T_2}(y) = P_2R$.

By a similar argument we may prove that the two monotone increasing functions $\rho_1(\lambda) = \| E_1(\lambda) x \|^2$ and $\rho_2(\lambda) = \| E_2(\lambda) y \|^2$ are mutually absolutely continuous with respect to each other. Hence, by Radon-Nikodym's theorem, there exists a Borel measurable non-negative function $F(\lambda)$ such that

$$\rho_1(\lambda) = \int_{-\infty}^{\lambda} F(\lambda)\, d\rho_2(\lambda), \quad \rho_2(\lambda) = \int F(\lambda)^{-1} d\rho_1(\lambda).$$

hence, if we put $y(x) = \int \sqrt{F(\lambda)}\, dE_2(\lambda)\, y$, we have

$$M_{T_2}(y(x)) = M_{T_2}(y) = P_2R, \quad \rho(\lambda) = \| E_1(\lambda) x \|^2 = \| E_2(\lambda) y(x) \|^2.$$

Thus it is easy to see that the isometric operator V demanded in (4) is given by

$$VF(T_1)x = F(T_2)\, y(x).$$

Remork. Our heorem may easily be extended to the case where T_1 and T_2 are normal operators.

In concluding this note I express my hearty thanks to Dr. Kiyosi Itô for the discussion of the result.